Black Holes, Minimal Surfaces, and Differential Geometry

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In this talk we will describe the beautiful relationship between the geometry of minimal surfaces in a Riemannian 3-manifold and the physical properties of black holes in a spacetime. The theorems we will discuss will have both geometric and physical interpretations. The physical interpretations usually appear to be statements which are "obvious," yet the rigorous proofs of these statements using differential geometry are very often quite challenging and have motivated the development of new and unexpected techniques.

A great example is the Penrose Conjecture in General Relativity. In 1973, Roger Penrose made a compelling physical argument that the mass contributed by a collection of black holes in a spacetime should be at least $\sqrt{A/16\pi}$, where A is the total surface area of the event horizons of the black holes. Assuming nonnegative energy density in the spacetime, he then conjectured the physically compelling inequality that the total mass of the spacetime $m \geq \sqrt{A/16\pi}$.

Geometrically, however, this inequality is highly nontrivial and, in its most general form, is still an open conjecture. A special case of the conjecture is known as the Riemannian Penrose Conjecture. In this case, we have added the assumption that the spacetime admits a totally geodesic space-like submanifold (M^3, g), which we can think of as a "$t = 0$" slice of the

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spacetime. In this setting, apparent horizons of black holes correspond to compact minimal surfaces of (M^3, g), and the assumption of nonnegative energy density implies that (M^3, g) has nonnegative scalar curvature. (M^3, g) is assumed to be "asymptotically flat," meaning that outside a compact set it is diffeomorphic to $\mathbf{R}^3 \backslash B_1(0)$, with the metric approaching the standard flat metric on $\mathbf{R}^3$ at infinity. The total mass m is a parameter related to the rate at which the manifold becomes flat at infinity. Penrose's argument about black holes then suggests that the value of this parameter m is at least $\sqrt{A/16\pi}$, where A is the total area of the collection of outermost compact minimal surfaces of (M^3, g), assuming (M^3, g) has nonnegative scalar curvature.

Gerhard Huisken and Tom Ilmanen proved the Riemannian Penrose Inequality for a single black hole in 1998 by using inverse mean curvature flows of surfaces in a 3-manifold. A different proof of the conjecture which works for any number of black holes will be sketched, and involves the definition of a new flow of metrics which preserves nonnegative scalar curvature.