Well Balanced Methods for Conservation Laws with Source Terms

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Cons. law with source terms (balance law)

In one space dimension:

$$q_t + f(q)_x = \psi(q)\sigma_x(x)$$

Note: if $\sigma(x) = x$ then source term is just $\psi(q)$.

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Goal: Compute accurate quasi-steady solutions, small perturbations of equilibria $q^e(x)$ satisfying

$$f(q^e)_x = \psi(q^e)\sigma_x(x).$$

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$$f(q^e)_x = \psi(q^e)\sigma_x(x).$$

Method is well balanced if $q^e(x)$ is exactly maintained by numerical method.

Fractional steps for a quasisteady problem

Alternate between solving homogeneous conservation law

$$q_t + f(q)_x = 0 \qquad (1)$$

and source term

$$q_t = \psi(q). \qquad (2)$$

When $q_t \ll f(q)_x \approx \psi(q)$:

- Solving (1) gives large change in q
- Solving (2) should essentially cancel this change.

Numerical difficulties:

- (1) and (2) are solved by very different methods. Generally will not have proper cancellation.
- Nonlinear limiters are applied to $f(q)_x$ term, not to small-perturbation waves. Large variation in steady state hides structure of waves.

Equilibrium solutions

Shallow water equations with bathymetry/topography $B(x)$:

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x(x)$$

Ocean-at-rest equilibrium:

$$u^e \equiv 0, \qquad h^e(x) + B(x) \equiv \bar{\eta} = \text{sea level.}$$

The hydrostatic pressure gradient $\frac{1}{2}g(h^2)_x$ balances the source term.

Stationary atmosphere with pressure gradient balancing gravity is similar.

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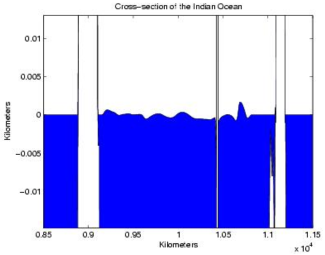
Flowing equilibria: Stationary solutions have $hu \equiv$ constant and

$$\left(hu^2 + \frac{1}{2}gh^2\right)_x + ghB_x(x) = 0.$$

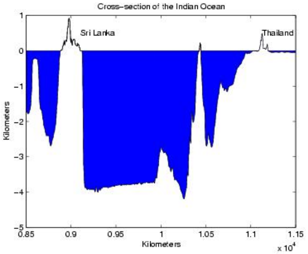
If solution is smooth then $E = \frac{1}{2}u^2 + g(h + B)$ is also constant in space.

Cross section of Indian Ocean & tsunami

Surface elevation
on scale of 10 meters:



Cross-section
on scale of kilometers:



Advection-decay example

$$q_t + uq_x = -q\sigma_x(x)$$

Solution advects with speed u and decays where $\sigma_x > 0$, grows where $\sigma_x < 0$.

Ex: $\sigma(x) = x \implies q(x,t) = e^{-t/u}q^0(x - ut)$

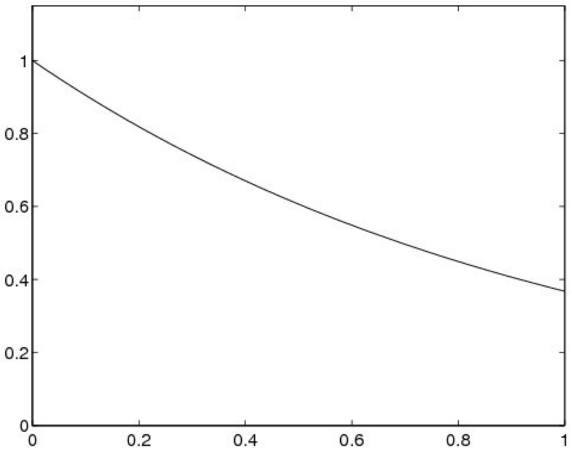
IBVP on $x \geq 0$ with $q(0,t) = \mu$ has

Equilibrium solution: $q^e(x) = \mu e^{-x/u}$.

Decaying exponential propagates to right but decays downwards and the two effects balance out.

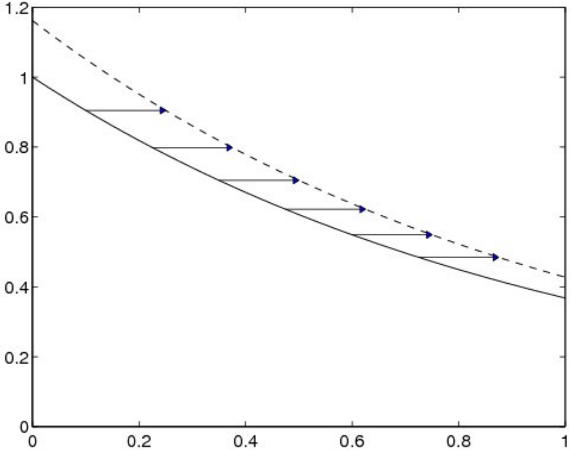
Advection-decay

Equilibrium solution



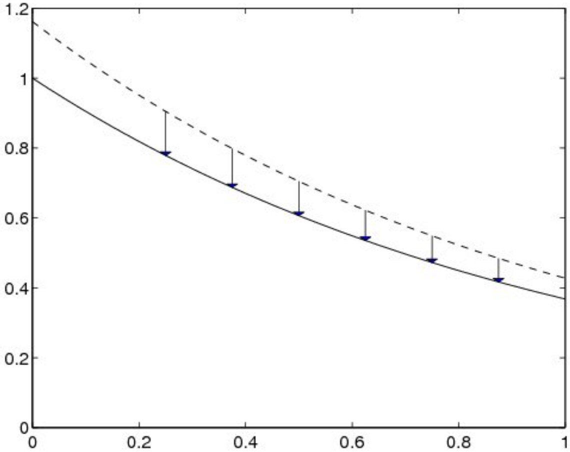
Advection-decay

Advection

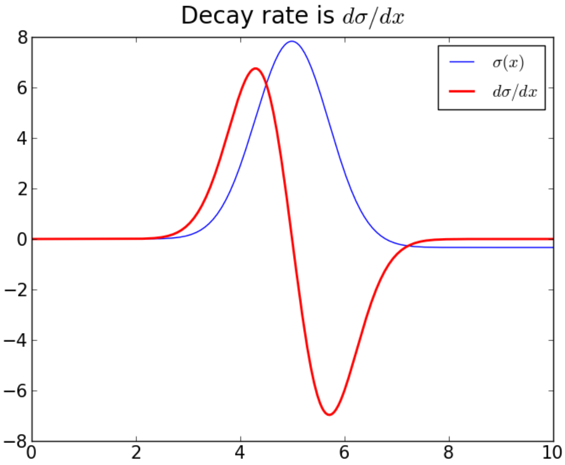


Advection-decay

Exponential decay



Advection-decay equation: $q_t + uq_x = -q\sigma_x(x)$



Demos...

Outline

- Wave propagation algorithms for $q_t + f(q)_x = 0$
  Riemann solver:
    Jump $Q_i - Q_{i-1}$ is split into waves
  Wave limiters for high resolution without oscillations.

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- Incorporate source term: $q_t + f(q)_x = \psi(q)\sigma_x$:
  $f(Q_i) - f(Q_{i-1}) - \Psi_{i-1/2}(\sigma_i - \sigma_{i-1})$ is split into waves

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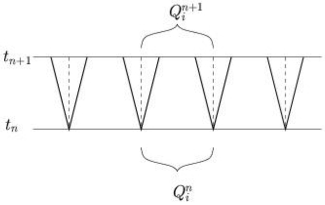
- Choice of $\Psi_{i-1/2}$: Path conservative approach

Godunov's Method for $q_t + f(q)_x = 0$

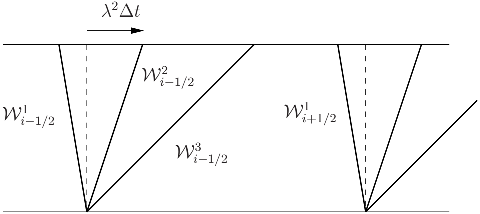


1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}^p_{i-1/2}$ and speeds $s^p_{i-1/2}$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(s_{i-1/2}^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(s_{i+1/2}^p)^-\mathcal{W}_{i+1/2}^p\right]$$

where

$$s^+ = \max(s,0), \qquad s^- = \min(s,0).$$

Note: Requires only waves and speeds.

Applicable also to hyperbolic problems not in conservation form.

For $q_t + f(q)_x = 0$, conservative if waves chosen properly,
e.g. using Roe-average of Jacobians.

Great for general software, but only first-order accurate (upwind method for linear systems).

Wave-propagation form of high-resolution method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(s_{i-1/2}^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(s_{i+1/2}^p)^-\mathcal{W}_{i+1/2}^p\right] - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

Correction flux:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{M_w}|s_{i-1/2}^p|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right)\widetilde{\mathcal{W}}_{i-1/2}^p$$

where $\widetilde{\mathcal{W}}_{i-1/2}^p$ is a limited version of $\mathcal{W}_{i-1/2}^p$ to avoid oscillations.

(Unlimited waves $\widetilde{\mathcal{W}}^p = \mathcal{W}^p \implies$ Lax-Wendroff for a linear system $\implies$ nonphysical oscillations near shocks.)

Summary of wave propagation algorithms

For $q_t + f(q)_x = 0$, the flux difference

$$\mathcal{A}\Delta Q_{i-1/2} = f(Q_i) - f(Q_{i-1})$$

is split into:

left-going fluctuation: $\mathcal{A}^-\Delta Q_{i-1/2}$, updates Q_{i-1},
right-going fluctuation: $\mathcal{A}^+\Delta Q_{i-1/2}$, updates Q_i,
Waves: $Q_i - Q_{i-1} = \sum \alpha^p r^p = \sum \mathcal{W}^p$
Often take $\mathcal{A}^\pm\Delta Q_{i-1/2} = \sum (s^p)^\pm \mathcal{W}^p$.

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f-wave formulation: Bale, RJL, Mitran, Rossmanith, SISC 2002

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In either case, limiters are applied to waves or f-waves for use in high-resolution correction terms.

Incorporating source term in f-waves

$q_t + f(q)_x = \psi(q)\sigma_x(x)$

Concentrate source at interfaces: $\Psi_{i-1/2}(\sigma_i - \sigma_{i-1})$

Split $f(Q_i) - f(Q_{i-1}) - (\sigma_i - \sigma_{i-1})\Psi_{i-1/2} = \sum_p \mathcal{Z}^p_{i-1/2}$

Use these waves in wave-propagation algorithm.

Steady state maintained:

If $\frac{f(Q_i)-f(Q_{i-1})}{\Delta x} = \Psi_{i-1/2}\frac{(\sigma_i-\sigma_{i-1})}{\Delta x}$ then $\mathcal{Z}^p \equiv 0$

Near steady state:

Deviation from steady state is split into waves and limited.

Incorporating source term in f-waves

$$q_t + f(q)_x = \psi(q)\sigma_x(x) \implies \Psi_{i-1/2}(\sigma_i - \sigma_{i-1})$$

Question: How to average $\psi(q)$ between cells to get $\Psi_{i-1/2}$?

For some problems (e.g. ocean-at-rest) can simply use arithmetic average.

$$\Psi_{i-1/2} = \frac{1}{2}(\psi(Q_{i-1}) + \psi(Q_i)).$$

Shallow water equations with bathymetry $B(x)$

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$$\begin{aligned} f(Q_i) - f(Q_{i-1}) - \Psi_{i-1/2}(B_i - B_{i-1}) &= \\ &= \left(\frac{1}{2}gh_i^2 - \frac{1}{2}gh_{i-1}^2\right) + \frac{g}{2}(h_{i-1} + h_i)(B_i - B_{i-1}) \\ &= \frac{g}{2}(h_{i-1} + h_i)((h_i + B_i) - (h_{i-1} + B_{i-1})) \\ &= 0 \quad \text{if } h_i + B_i = h_{i-1} + B_{i-1} = \bar{\eta}. \end{aligned}$$

Advection-decay example

$$q_t + uq_x = -q\sigma_x(x).$$

on $0 \le x \le 10$ with

$$\sigma(x) = Ae^{-(x-5)^2} + B(\tanh(x-5)+1)$$

Equilibrium solution:

$$q(x,0) = q^e(x) = e^{-\sigma(x)/u}.$$

Note: σ is nearly flat at boundaries, increasing and then decreasing in domain.

So solution advects with speed u and decays to nearly zero, then grows again.

Demos...

Use path conservative approach

Rewrite $q_t + f(q)_x = \psi(q)\sigma_x(x)$ as

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This is a nonconservative hyperbolic system $w_t + B(w)w_x = 0$ with:

$$w = \begin{bmatrix} q \\ \sigma \end{bmatrix}, \qquad B(w) = \begin{bmatrix} f'(q) & -\psi(q) \\ 0 & 0 \end{bmatrix}.$$

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Eigenvalues of B are those of A and $\lambda = 0$.

Assume "nonresonant" case: 0 is not an eigenvalue of A.

Nonconservative hyperbolic problems

Now consider

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Correct discontinuous (weak) solutions harder to define.

Vanishing viscosity approach: add diffusive term of order ϵ,

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Note that if q is discontinuous then q_x is a delta function with support at points where $A(q)$ is discontinuous.

Dal Maso, LeFloch, Murat (DLM) theory

$$q_t + A(q)q_x = 0.$$

Define nonconservative product in terms of Borel measures related to family of paths in state space.

$$q(s) = \Phi(q_l, q_r, s)$$

is a path from q_l to q_r parameterized by $s \in [0, 1]$.

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$$\xi(q_r - q_l) = \int_0^1 A(q(s))q'(s)\, ds.$$

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If $A(q)q_x = f(q)_x$ (conservative) then

$$\int_0^1 A(q(s))q'(s)\,ds = f(q_r) - f(q_l)$$

independent of the choice of path.

Path conservative numerical methods

Given $Q_{i-1},\ Q_i$, want to determine $\mathcal{A}^-\Delta Q_{i-1/2},\ \mathcal{A}^+\Delta Q_{i-1/2}$ to increment cells on either side:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2})$$

For a conservation law: Conservative method if

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Path conservative method (for a given family of paths Φ):
Choose $\mathcal{A}^-\Delta Q_{i-1/2},\ \mathcal{A}^+\Delta Q_{i-1/2}$ so that

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For any path $w(s)$ in state space we obtain

$$\int_0^1 B(w(s))w'(s)\,ds = \int_0^1 \begin{bmatrix} f'(q(s)) & -\psi(q(s)) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q'(s) \\ \sigma'(s) \end{bmatrix} ds$$

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If $\sigma(s) = \sigma_{i-1} + s(\sigma_i - \sigma_{i-1})$, then $\sigma'(s) = \sigma_i - \sigma_{i-1}$.

$\implies$ Recover f-wave modification with particular average of ψ.

Use path conservative approach

For any path $w(s)$ in state space we obtain

$$\int_0^1 B(w(s))w'(s)\,ds = \int_0^1 \begin{bmatrix} f'(q(s)) & -\psi(q(s)) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q'(s) \\ \sigma'(s) \end{bmatrix} ds$$

$$= \begin{bmatrix} \int_0^1 f'(q(s))q'(s) - \psi(q(s))\sigma'(s)\,ds \\ 0 \end{bmatrix}.$$

No increment to σ as expected. Increment to Q is:

$$\mathcal{A}\Delta Q_{i-1/2} = f(Q_i) - f(Q_{i-1}) - \int_0^1 \psi(q(s))\sigma'(s)\,ds.$$

If $\sigma(s) = \sigma_{i-1} + s(\sigma_i - \sigma_{i-1})$, then $\sigma'(s) = \sigma_i - \sigma_{i-1}$.

$\Longrightarrow$ Recover f-wave modification with particular average of ψ.

Can do better: Choose path in state space to simplify integral and guarantee well balanced.

Equilibrium path

Consider path $(Q_{i-1}, \sigma_{i-1}) \quad \longrightarrow \quad (\hat{Q}_i, \sigma_i) \quad \longrightarrow \quad (Q_i, \sigma_i)$
parameterized by s going from 0 to 1/2 to 1, say.

First part: σ varies, q remains in equilibrium
(Source term balances $f'(q(s))q'(s)$)

Second part: σ constant.

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$$\int_0^1 f'(q(s))q'(s) - \psi(q(s))\sigma'(s)\, ds$$

$$= \int_{1/2}^1 f'(q(s))q'(s) - \psi(q(s))\sigma'(s)\, ds$$

$$= \int_{1/2}^1 f'(q(s))q'(s)\, ds$$

$$= f(Q_i) - f(\hat{Q}_i)$$

Equilibrium path

Example: $q_t + uq_x = -q\sigma_x(x)$.

For $u > 0$ the Riemann solution for W_{i-1}, W_i has two waves:
speed 0: jump from (Q_{i-1}, σ_{i-1}) to $(\hat{Q}_i, \sigma_i)$,
speed u: jump from $(\hat{Q}_i, \sigma_i)$ to (Q_i, σ_i).

where $\hat{Q}_i = Q_{i-1}e^{-\Delta x/u}$.

So

$$\int_0^1 f'(q(s))q'(s) - \psi(q(s))\sigma'(s)\,ds = f(Q_i) - f(\hat{Q}_i)$$
$$= u(Q_i - Q_{i-1}e^{-\Delta x/u})$$

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$$\int_0^1 f'(q(s))q'(s) - \psi(q(s))\sigma'(s)\,ds = f(Q_i) - f(\hat{Q}_i)$$
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This is the quantity we split into waves.
Equals zero when solution is in equilibrium.   Demos...

A few references

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