Vanishing Viscosity Limits for a Class of Circular Pipe Flows

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Navier-Stokes equations

$$(1) \qquad \frac{\partial u^\nu}{\partial t} + \nabla_{u_\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu + F^\nu, \quad \operatorname{div} u^\nu = 0.$$

on $\mathbb{R}^+ \times \Omega, \quad \Omega = D \times \mathbb{R}, \quad D = \{x \in \mathbb{R}^2 : |x| < 1\}, \quad z \in \mathbb{R}.$

$$F^\nu(t, x, z) = (0, f^\nu(t)).$$

No slip boundary condition:

$$u^\nu(t, x, z) = \Bigl(\frac{\alpha(t)}{2\pi} x^\perp, \beta(t)\Bigr), \quad |x| = 1,\ z \in \mathbb{R},\ t > 0.$$

Special initial condition (in addition to $\operatorname{div} u_0 = 0,\ u_0 \parallel \partial\Omega$)

$u^\nu(0, x, z) = u_0(x) = (v_0(x), w_0(x)),$

v_0 rotationally symmetric: $v_0(x) = s_0(|x|)x^\perp \ (+s_1(|x|)x).$

$$v_0(R_\theta x) = R_\theta v_0(x) \ \text{ plus } \operatorname{div} v_0 = 0 \ \text{ plus } v_0 \parallel \partial D.$$

These conditions imply $v_0(x) = s_0(|x|)x^\perp$. Another characterization:

$$v_0(\Phi_\omega x) = -\Phi_\omega v_0(x).$$

$$R_\theta = \text{ rotation}, \quad \Phi_\omega = \text{ reflection}.$$

Lemma 1. *In such a case,*

$$u^\nu = (v^\nu(t,x), w^\nu(t,x)), \quad p^\nu = p^\nu(t,x).$$

NOTE.

$$\nabla_{u^\nu} u^\nu = (\nabla_{v^\nu} v^\nu, \nabla_{v^\nu} w^\nu), \quad \operatorname{div} u^\nu = \operatorname{div} v^\nu.$$

Hence (1) becomes

$$\partial_t v^\nu + \nabla_{v^\nu} v^\nu + \nabla p^\nu = \nu \Delta v^\nu, \quad \operatorname{div} v^\nu = 0 \tag{2}$$

$$\partial_t w^\nu + \nabla_{v^\nu} w^\nu = \nu \Delta w^\nu + f^\nu(t). \tag{3}$$

Boundary conditions:

$$v^\nu = \frac{\alpha(t)}{2\pi} x^\perp \ \text{ on } \ \partial D, \quad w^\nu = \beta(t) \ \text{ on } \ \partial D.$$

Euler equations

$$\frac{\partial u^0}{\partial t} + \nabla_{u^0} u^0 + \nabla p^0 = F^0, \quad \operatorname{div} u^0 = 0.$$

$$\text{Boundary condition} \;\; u^0 \parallel \partial\Omega.$$

Initial condition $u^0(0, x, z) = (v_0(x), w_0(x))$, same as before.

Then $u^0(t, x, z) = (v^0(t, x), w^0(t, x))$, with

$$\partial_t v^0 + \nabla_{v^0} v^0 + \nabla p^0 = 0, \quad \operatorname{div} v^0 = 0, \tag{5}$$

$$\partial_t w^0 + \nabla_{v^0} w^0 = f^0(t). \tag{6}$$

Boundary condition:

$$v^0 \parallel \partial D \;\; \text{for} \;\; t > 0, \; x \in \partial D.$$

Lemma 2. *$v^0(t, x) \equiv v_0(x)$ is steady solution to (5).*

Proof. One checks that

$$\nabla_{v_0} v_0 = -s_0(|x|)^2 x = -\nabla p_0(x),$$

where

$$p_0(x) = \tilde{p}_0(|x|), \quad \tilde{p}_0(r) = -\int_r^1 \rho s_0(\rho)^2 \, d\rho.$$

Then (6) becomes

$$\partial_t w^0 + \nabla_{v_0} w^0 = f^0(t). \quad \text{(No boundary condition)},$$

with solution

$$w^0(t,x) = w_0(\mathcal{F}_{v_0}^{-t}(x)) + \int_0^t f^0(s)\, ds.$$

Here $\mathcal{F}_{v_0}^{-t}$ is the backwards flow of $\overline{D}$ generated by v_0.

Expanded verification of proof of Lemma 2

Assume $v(x) = s(|x|)x^\perp = s(|x|)(-x_2, x_1)$. Then

$$\nabla_v v = -x_2 s(|x|)\partial_1 v + x_1 s(|x|)\partial_2 v.$$

Now,

$$\partial_1 v = s(|x|)(0,1) + s'(|x|)\frac{x_1}{|x|}(-x_2, x_1)$$
$$\partial_2 v = s(|x|)(-1,0) + s'(|x|)\frac{x_2}{|x|}(-x_2, x_1).$$

Hence

$$\begin{aligned}\nabla_v v &= -x_2 s^2(0,1) - x_2 s s'\frac{x_1}{|x|}(-x_2, x_1)\\ &\quad + x_1 s^2(-1,0) + x_1 s s'\frac{x_2}{|x|}(-x_2, x_1)\\ &= -s(|x|)^2(x_1, x_2)\\ &= -s(|x|)^2 x.\end{aligned}$$

Also,

$$\partial_j \tilde{p}((x_1^2 + x_2^2)^{1/2}) = \tilde{p}'(|x|)\frac{x_j}{|x|},$$

so

$$p(x) = \tilde{p}(|x|) \Longrightarrow \nabla p(x) = \tilde{p}'(|x|)\frac{x}{|x|} = \frac{\tilde{p}'(|x|)}{|x|}x.$$

Hence

$$\nabla_v v = -\nabla p, \quad \tilde{p}'(r) = rs(r)^2, \text{ i.e.,}$$
$$\tilde{p}(r) = -\int_r^1 \rho s(\rho)^2\, d\rho.$$

Problem: Study how

$$v^\nu \longrightarrow v^0 \equiv v_0 \;\text{ and }\; w^\nu \longrightarrow w^0 \;\text{ as }\; \nu \searrow 0. \tag{8}$$

Literature on first case: Matsui, Wang, Bona-Wu, [LMN], ...

Proposition 3. *The solution to (2) is circularly symmetric for all $t > 0$, and hence solves the linear equation*

$$\frac{\partial v^\nu}{\partial t} = \nu \Delta v^\nu, \quad v^\nu\Big|_{\mathbb{R}^+ \times \partial D} = \frac{\alpha(t)}{2\pi} x^\perp.$$

Hence

$$v^\nu(t) = e^{\nu t \Delta} v_0 + \int_0^t (I - e^{\nu(t-s)\Delta}) f_1 \, d\alpha(s),$$

where

$$f_1(x) = \frac{x^\perp}{2\pi}.$$

Corollary 4. *Take a Banach space* $\mathfrak{X}$ *such that* $v_0, f_1 \in \mathfrak{X}$ *and* $\{e^{t\Delta} : t \geq 0\}$ *is strongly continuous on* $\mathfrak{X}$*. Then, as* $\nu \searrow 0$*,*

$$\|v^\nu(t) - v_0\|_{\mathfrak{X}} \longrightarrow 0,$$

uniformly in $t \in [0, T_0]$*.*

$\{e^{t\Delta} : t \geq 0\}$ strongly continuous semigroup on

$$\begin{aligned}
&H^{s,p}(D), \quad s \geq 0, \ \ p \in [1, \infty), \ \ sp < 1, \\
&\mathcal{V}^k(D), \ \ \text{weighted } b\text{-Sobolev space}, \\
&C_*(D) = \{f \in C(\overline{D}) : f|_{\partial D} = 0\}, \\
&H_0^{1,p}(D), \quad 1 \leq p < \infty, \\
&C_*^1(\overline{D}) = \{f \in C^1(\overline{D}) : f|_{\partial D} = 0\}.
\end{aligned}$$

Fine analysis of $e^{t\Delta}f$, given $f \in C^\infty(\overline{\Omega})$.

$$\overline{\Omega} \subset M, \quad f \text{ extends to } \tilde{f}.$$

$$\begin{aligned} e^{t\Delta}f &= e^{t\Delta}\tilde{f} - U(t,x) \\ &= f(x) + \sum_{k=1}^{N} \frac{t^k}{k!}\Delta^k f(x) + \widehat{R}_N(t,x) - U(t,x). \end{aligned}$$

$$(\partial_t - \Delta)U = 0 \text{ on } \mathbb{R}\times\Omega, \quad U(t) = 0 \text{ for } t < 0,$$
$$U(t,\cdot)\Big|_{\partial\Omega} = \chi_{\mathbb{R}^+}(t)e^{t\Delta}\tilde{f}\Big|_{\partial\Omega}.$$

$$e^{t\Delta}f(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-s^2/4t}\cos s\sqrt{-\Delta}f(x)\,ds.$$

Ditto with $\tilde{f}$ in place of f.

$$\cos s\sqrt{-\Delta}f = \cos s\sqrt{-\Delta}\tilde{f} - V(s,x).$$

$$(\partial_s^2 - \Delta)V = 0 \text{ on } \mathbb{R}\times\Omega, \quad V(s) = 0 \text{ for } s < 0,$$
$$V(s,\cdot)\Big|_{\partial\Omega} = \chi_{\mathbb{R}^+}(s)\cos s\sqrt{-\Delta}\tilde{f}\Big|_{\partial\Omega}.$$

Hence

$$U(t,x) = \frac{2}{\sqrt{4\pi t}}\int_0^\infty e^{-s^2/4t}V(s,x)\,ds.$$

For $V(s,x)$, we have a progressing wave expansion:

$$V(s,x) = \sum_{j=0}^{N} b_j(x)(s - \varphi(x))_+^j + R_N(s,x).$$

Here,

$$\varphi(x) = \mathrm{dist}(x, \partial\Omega).$$

In particular,

$$\Omega = D \Longrightarrow \varphi(x) = 1 - |x|.$$

Main conclusion:

$$\begin{aligned} e^{t\Delta} f(x) \sim & \sum_{k\ge 0} \frac{t^k}{k!} \Delta^k f(x) \\ & - \sum_{j\ge 0} 2 b_j(x)(4t)^{j/2} E_j\Big(\frac{\varphi(x)}{\sqrt{4t}}\Big), \end{aligned}$$

where

$$E_j(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2} (s-y)^j \, ds.$$

Note

$$b_j\big|_{\partial\Omega} = 0 \quad \text{when} \quad j \quad \text{is odd.}$$

$$b_0\big|_{\partial\Omega} = f\big|_{\partial\Omega}, \quad E_0(0) = \frac{1}{2}.$$

Second part of (8): How $w^\nu \longrightarrow w^0$.

$$(*)\qquad \begin{aligned} \partial_t w^\nu &= (\nu\Delta - X_\nu)w^\nu + f^\nu(t) \quad \text{on} \quad \mathbb{R}^+ \times D, \\ w^\nu(t) &= \beta(t) \quad \text{on } \mathbb{R}^+ \times \partial D, \\ w^\nu(0) &= w_0(x). \end{aligned}$$

$$X_\nu g = \nabla_{v^\nu} g = s^\nu(t, |x|)\frac{\partial g}{\partial \theta}.$$

$$\partial_t w^0 = -Xw^0 + f^0(t), \quad w^0(0,x) = w_0(x).$$

$$Xg = \nabla_{v_0} g = s_0(|x|)\frac{\partial g}{\partial \theta}.$$

Separate 2 singular perturbation mechanisms:

$$\partial_t w^\nu = (\nu\Delta - X)w^\nu + (X - X_\nu)w^\nu + f^\nu(t). \tag{9}$$

Set $W^\nu(t,x) = w^\nu(t,x) - \beta(t)$, which solves

$$\begin{aligned} \partial_t W^\nu &= (\nu\Delta - X)W^\nu + (X - X_\nu)W^\nu + g^\nu(t), \\ W^\nu(t,x) &= 0 \ \text{ on } \ \mathbb{R}^+ \times \partial D, \\ W^\nu(0,x) &= W_0(x) = w_0(x) - \beta(0). \end{aligned} \tag{10}$$

Here,

$$g^\nu(t) = f^\nu(t) - \beta'(t).$$

Note

$$w^\nu - w^0 = W^\nu - W^0,$$

where

$$\partial_t W^0 = -XW^0 + g^0(t), \quad W^0(0,x) = W_0(x). \tag{10A}$$

Apply Duhamel's formula to (10)–(10A), to get

$$w^\nu(t,x) - w^0(t,x) = R_1(\nu,t,x) + R_2(\nu,t,x) + R_3(\nu,t,x),$$

with

$$\begin{aligned}
R_1(\nu,t,x) &= e^{t(\nu\Delta - X)}W_0 - e^{-tX}W_0, \\
R_2(\nu,t,x) &= \int_0^t [f^\nu(s) - f^0(s)]\, ds \\
&\quad + \int_0^t (e^{\nu(t-s)\Delta}1 - 1) g^\nu(s)\, ds \\
R_3(\nu,t,x) &= \int_0^t e^{(t-s)(\nu\Delta - X)}(s_0 - s^\nu)\frac{\partial w^\nu}{\partial \theta}\, ds.
\end{aligned}$$

Precise analysis of $R_2(\nu,t,x)$ easy, by fine analysis of $e^{t\Delta}f$.

Analysis of $R_3(\nu, t, x)$.

Take $Z^\nu = \partial_\theta w^\nu$. Apply ∂_θ to (*). Since ∂_θ commutes with Δ and with X_ν, we get

$$\frac{\partial Z^\nu}{\partial t} = (\nu\Delta - X_\nu)Z^\nu, \quad Z^\nu\Big|_{\mathbb{R}^+ \times \partial D} = 0, \quad Z^\nu(0) = \partial_\theta w_0.$$

Maximum principle gives

$$\|\partial_\theta w^\nu(s)\|_{L^\infty(D)} \le \|\partial_\theta w_0\|_{L^\infty(D)}.$$

Radial symmetry gives

$$e^{(t-s)(\nu\Delta - X)}|s_0 - s^\nu| = e^{\nu(t-s)\Delta}|s_0 - s^\nu|.$$

Since $e^{t(\nu\Delta - X)}$ is positivity preserving, we get

$$|R_3(\nu, t, x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta}|s_0 - s^\nu|\, ds$$

$\Longrightarrow$ good estimates, including boundary layer thickness (but no boundary layer resolution)

Next task: analyze $R_1(\nu, t, x)$, given $W_0 \in C^\infty(\overline{D})$.

Analysis of $e^{t(\nu\Delta - X)} W_0, \quad W_0 \in C^\infty(\overline{\Omega})$.

$$\overline{\Omega} \subset M, \quad W_0 \ \text{ extends to } \ \widetilde{W}_0 \in C^\infty(M),$$
$$X \ \text{ extends to } C^\infty \text{ vector field.}$$

Set

$$v_\nu = e^{tX} e^{t(\nu\Delta - X)} W_0,$$
$$V_\nu = e^{tX} e^{t(\nu\Delta - X)} \widetilde{W}_0.$$

$$\frac{\partial}{\partial t} v_\nu = \nu e^{tX} \Delta e^{-tX} v_\nu = \nu L(t) v_\nu, \tag{11}$$

$$\frac{\partial}{\partial t} V_\nu = \nu e^{tX} \Delta e^{-tX} V_\nu = \nu L(t) V_\nu, \tag{12}$$

where

$$L(t) = e^{tX} \Delta e^{-tX}.$$

We have

$$v_\nu(t,x) = V_\nu(t,x) - u_\nu(t,x),$$

where u_ν is found by the method of layer potentials (see below).

Solving (12) for V_ν leads to

$$V_\nu(t,x) = \int_M V_\nu(s,y) H(\nu,s,t,x,y)\, dV_s(y),$$

for $0 \le s < t$, where $H(\nu,s,t,x,y)$ is obtained by a variant of heat kernel asymptotics, also connected to semiclassical asymptotics.

Formulas for $H(\nu,s,t,x,y)$.

$$S_\nu^t f(x) = (2\pi)^{-n/2} \int a(\nu,t,x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

$$a(\nu,t,x,\xi) \sim \sum_{j\ge 0} a_j(\nu,t,x,\xi).$$

The a_j satisfy transport equations. First transport equation:

$$\frac{\partial a_0}{\partial t} = -\nu G(t,x,\xi) a_0, \quad a_0(\nu,0,x,\xi) = 1,$$
$$G(t,x,\xi) = -L_2(t,x,\xi) \quad \text{(symbol of } L(t)\text{)}.$$

Solution

$$a_0(\nu,t,x,\xi) = e^{-\nu t H(t,x,\xi)}, \quad H(t,x,\xi) = \frac{1}{t}\int_0^t G(s,x,\xi)\, ds.$$

Take Fourier transform to get integral kernels:

$$H(\nu, s, t, x, y) = g(s, y)^{-1/2} K(\nu, s, t, x, x - y),$$
$$K \sim \sum_{j \geq 0} K_j(\nu, s, t, x, z),$$

$$K_0(\nu, s, t, x, z) = \big(4\pi\nu(t - s)\big)^{-n/2} \det \mathcal{G}(s, t, x)^{1/2}$$
$$e^{-\mathcal{G}(s,x,x)z \cdot z / 4\nu(t-s)}.$$

$$H_0(\nu, s, t, x, y) = g(s, y)^{-1/2} K_0(\nu, s, t, x - y).$$

$$\Omega = D \Longrightarrow n = 2.$$

Layer potential methods give u_ν.

$$u_\nu(t,x) = \mathcal{D}_\nu h_\nu(t,x), \tag{13}$$

where

$$\mathcal{D}_\nu h(t,x) = \nu \int_0^t \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, y)\, dS_s(y)\, ds.$$

Trace formula:

$$\mathcal{D}_\nu h\Big|_{\mathbb{R}\times\partial D} = \Big(\frac{1}{2}I + \nu N_\nu\Big)h.$$

In (13), h_ν solves

$$\Big(\frac{1}{2}I + \nu N_\nu\Big)h_\nu = g_\nu = \chi_{\mathbb{R}^+}(t)V_\nu(t,x), \quad x \in \partial D.$$

Note that

$$\Big\| \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, \cdot)\Big\|_{L^1(D)} \le C((t-s)\nu)^{-1/2}, \quad x \in \partial D,$$

so

$$\|\nu N_\nu h\|_{L^\infty(I\times\partial D)} \le C(I)\nu^{1/2}\|h\|_{L^\infty(I\times\partial D)}.$$

Hence

$$\begin{aligned} h_\nu &= 2(I + 2\nu N_\nu)^{-1} g_\nu \\ &= 2(I - 2\nu N_\nu + 4\nu^2 N_\nu^2 - \cdots)g_\nu. \end{aligned}$$

Proposition 5. *Set*

$$\mathcal{D}^0_\nu h(t,x) = \nu \int_0^t \int\limits_{\partial D} h(s,y) \frac{\partial H_0}{\partial n_{s,y}}(\nu,s,t,x,y)\, dS_s(y)\, ds.$$

Then, if $v_0, w_0 \in C^\infty(\overline{D})$,

$$\|R_1(\nu,\cdot,\cdot) + 2e^{-tX}\mathcal{D}^0_\nu W^b_0\|_{L^\infty(I\times D)} \leq C(I)\nu^{1/2}\|W_0\|_{C^{1+\delta}(\overline{D})},$$

where $W^b_0 = \chi_{\mathbb{R}^+}(t) W_0|_{\partial D}$.