Accurate 2&3 Dimensional Interpolation Methods

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Designed for particle mesh (PM) methods

Can be used to interpolate fns that satisfy homogeneous elliptic PDEs, including ones with discontinuous coefficients
$\Longrightarrow$ Calculations on regions with different dielectrics

Methods can be used on other functions
$\Longrightarrow$ 2nd order accuracy

Particle Mesh Methods

Finite diff. methods for calculating particle interactions (field calc. on charged particles, vortex calc.)

$\Longrightarrow$ Less expensive and smoother solutions

1) Spread charge to the mesh W_q^s

2) Solve for potential $U_{i,j,k}$, Δ_h^{-1}

3) Difference potential $\Longrightarrow$ field

4) Interpolate field onto particles W_q^i

Fields singular $\Longrightarrow$ nearby interactions computed directly: P^3M (direct and mesh methods)

$W^s = W^i$, depend on force F:

$F(p,q) =$ force between particles at p, q,

$$LF = \delta(p-q), L_h = L + O(h^k)$$
$$LG = \delta(p-q)$$

$\Longrightarrow W^s = W^i = \frac{1}{c} L^h G$

$c = \sum W^s$

Coulomb forces $\Longrightarrow$

2D: $W^i = \Delta_h \frac{1}{2\pi} \log(|p-q|)$

3D: $W^i = \Delta_h \frac{1}{4\pi} \frac{1}{|p-q|}$

Anderson (1986), Colella(2007): homogeneous media
Mayo(1985) : inhomogeneous media

$W^s = L_h G \Longrightarrow$ accurate potential evaluation

$W^i = L_h G \Longrightarrow$ accurate field evaluation in P^3M calc.

L_h higher accurate $\Longrightarrow$ more accurate interpolation

$$Iv = \sum W^s(p,q)v(p) = \int (Gv_n - vG_n)ds = v(q)$$

Coulomb interactions in 2D:

$W^i = \Delta_h^5 G \Longrightarrow$ Midpoint rule

$W^i = \Delta_h^9 G \Longrightarrow$ Simpson's rule

Similar results for other operators in 2 & 3D

$W^i = W^s \Longrightarrow$ Schemes are Momentum Conserving

No self forces

Forces between particles are equal and opposite

Conservative methods conserve current in electron flow calculations

Nonconservative schemes can allow exponentially unstable motion

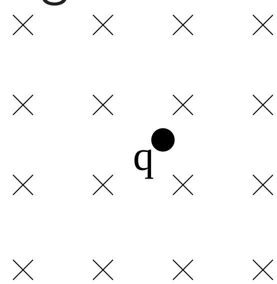
Particle Mesh method $\Longrightarrow$ field evaluation fast:

m particles, n^3 mesh $\Longrightarrow O(m + n^3 \log n)$

Only need to know G in a region around each particle $\Longrightarrow$ can solve problems on regions with many dielectrics layers.

Interpolating Harmonic Functions in 2D

$v(x_i, y_j)$ given on grid with uniform width h



$W^s(p,q) = \frac{1}{c}\Delta_h \frac{1}{2\pi}\log(|p-q|)$

$\Longrightarrow Iv(q) = \frac{1}{c}\sum v(p)\Delta_h \frac{1}{2\pi}\log(|p-q|)$

$$c \approx 1$$

Accurate potential evaluation:

In PM method $U_h = \Delta_h^{-1}(W^s)$

$U(p,q) = \frac{1}{2\pi}\log(|p-q|)$

Compute $\Delta_h \frac{1}{2\pi}\log(|p-q|)$ all p

$\Rightarrow U_h = U(p,q)$

$W^s = \Delta_h\left(\frac{1}{2\pi}\log(|p-q|)\right) \to 0$ as $|p-q| \to \infty \Longrightarrow$

U_h still accurate if $W^s \neq 0$ in small neighborhood of q

Compute W^s at more points, Δ_h higher accurate $\Longrightarrow$ U_h more accurate

$\sum_p W^i(p,q) \approx 1$, indep. of h:

$$\sum_p W^i(p,q) \approx \int_C G_n dS = 1$$

C curve around interpolation points.

$\Delta_h = \Delta_h^9 \leftrightarrow$ quad. rule = Simpson's rule,
Odd derivatives of integrand nearly cancel at endpts.

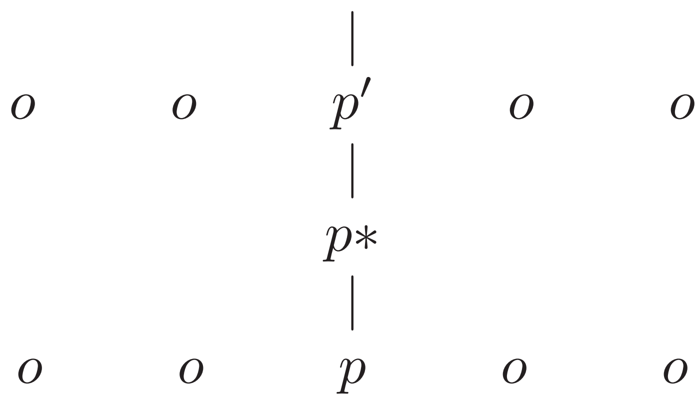
$\Delta_h = \Delta_h^5$:
Weights centered diff. $\Rightarrow \sum_p W(p,q)$
cancel except at edge of interpolation region

$*$	$*$	$*$	$*$	$*$	$*$
$*$	$\circ$	$\circ$	$\circ$	$\circ$	$*$
$*$	$\circ$	q		$\circ$	$*$
$*$	$\circ$			$\circ$	$*$
$*$	$\circ$	$\circ$	$\circ$	$\circ$	$*$
$*$	$*$	$*$	$*$	$*$	$*$

$\circ B_1$
$*B_2$

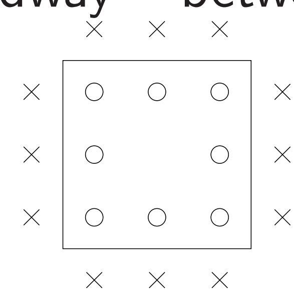
$\sum_p W(p,q) = \sum_{p \epsilon B_1} G(p,q) - G(p',q)$

$(G(p,q) - G(p',q)) \approx hG_n(p*)$



$\Longrightarrow \sum W(p,q) \approx \int_C G_n dS$

C = curve midway between B_1 and B_2



$\Longrightarrow \sum W(p,q) =$ midpt rule approx. to flux $+ O(h^3)$.

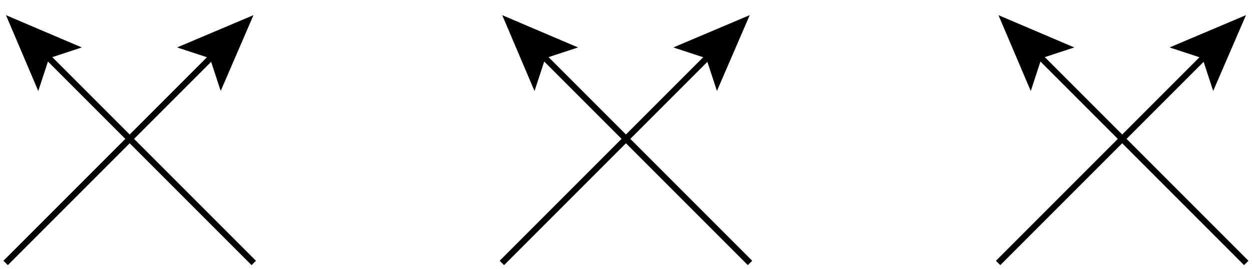
Fourth Order Accurate Discrete Laplacian

$$6h^2\Delta_9 = \begin{pmatrix} 1 & 4 & 1 \\ 4 & -20 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

$$= 4\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} + \begin{pmatrix} 1 & & 1 \\ & -4 & \\ 1 & & 1 \end{pmatrix}$$

$\sum W(p,q)$ contains differences of G in oblique directions directions as well as in coordinate directions.

$$u = \frac{y-x}{\sqrt{2}}, t = \frac{y+x}{\sqrt{2}}$$



Mixed derivatives from adjacent points $\approx G_n$

$$G_u + G_t = \sqrt{2}G_n$$

Coef. size = 1/2 coef. size exactly between pts.

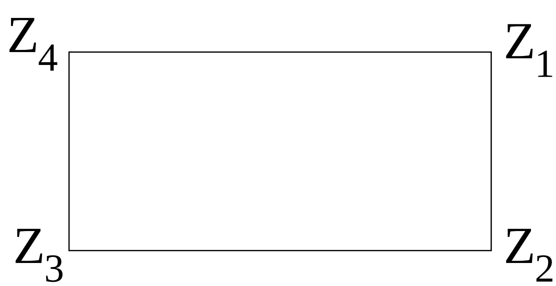
$$\begin{matrix} 1 & 4 & 2 & 4 & 2 & 4 & 1 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{matrix}$$

$\sum W(p, q) \approx \int_C G_n dS$, Simpson's rule, $h/2$.
G harmonic $\Longrightarrow$ $O(h^3)$ and certain higher order error terms cancel.

Euler MacLauren: $E_h = |\sum W(p, q) - 1|$

$$= \sum_{i=1}^{4} \frac{h^4}{2880} \left[\frac{\partial^3 G_n}{\partial s^3}(z_{i+1}, q) - \frac{\partial^3 G_n}{\partial s^3}(z_i, q) \right]$$

$$+ \sum \frac{h^6}{32.1512} \left[\frac{\partial^5 G_n}{\partial s^5}(z_{i+1}, q) - \frac{\partial^5 G_n}{\partial s^5}(z_i, q) \right] + O(h^8)$$



Region almost symmetric $\Longrightarrow$ error small.

3D Calculations

$$W(p,q) = \frac{1}{c}\Delta_h \frac{1}{4\pi r}$$

$\Delta_h = 27$ point approx (Lynch, 1977):

$$\Delta_h^{27} U(q) =$$
$$\frac{14\sum_{r^2=h^2} U(p) + 3\sum_{r^2=2h^2} U(p) + \sum_{r^2=3h^2} U(p) - 128U(p)}{30h^2}$$

$r = |p - q|$

$\sum W(p,q)$ is a 4th order accurate approx. to $\int_C G_n dS$

$2D$ quadrature formula for evaluating flux:

$$\frac{1}{30}\begin{pmatrix} 1 & 3 & 1 \\ 3 & 14 & 3 \\ 1 & 3 & 1 \end{pmatrix}$$

Linear combination of 2 $O(h^4)$ accurate quadrature formulas $\Longrightarrow$ $O(h^4)$ accurate

$$\frac{1}{30}\begin{pmatrix} 1 & 3 & 1 \\ 3 & 14 & 3 \\ 1 & 3 & 1 \end{pmatrix} = A\frac{1}{24}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 12 & 2 \\ 1 & 2 & 1 \end{pmatrix} + (1-A)\frac{(1)}{6}\begin{pmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{pmatrix}$$

$A = 4/5$.

$$S_q = \sum_p W(p,q)$$

2D:

Δ_h^9
2 rings : $|S_q - 1| < 5.1E-04$
4 rings : $|S_q - 1| < 7.9E-07$
8 rings : $|S_q - 1| < 1.2E-08$
16 rings : $|S_q - 1| < 1.9E-10$

Δ_h^5
2 rings : $|S_q - 1| < 1.4E-02$
4 rings : $|S_q - 1| < 3.3E-03$
8 rings : $|S_q - 1| < 8.3E-04$

3D:
2 rings : $|S_q - 1| < 1.7E-03$
4 rings : $|S_q - 1| < 2.3E-06$

Accuracy of Interpolation Formula

$$\int_D \int v \Delta u dV = \int_D \int u \Delta v dV + \int_{\partial D} (u_n v - v_n u)\, dS$$

$$\sum \sum v_i \Delta_h u_i = \sum \sum u_i \Delta_h v_i + \sum ((u_1 - u_0)v_i - u_i(v_1 - v_0))$$

$u = G, \Delta_h$ 4th order acc. $\Longrightarrow$

$$\sum \sum v_i \Delta_h G = cIv(q)$$

$$\sum \sum G_i \Delta_h v_i = \begin{cases} O(h^4) & \text{if } \Delta v = 0 \\ \int \int G \Delta v dV + O(h^2) & \text{if } \Delta v \neq 0 \end{cases}$$

$\sum ((u_1 - u_0)v_i - u_i(v_1 - v_0)) \approx \int (vG_n - v_n G) dS$
Simpson's rule

$$= v(p) \text{ if } \Delta v = 0$$

So

$$Iv(p) = \begin{cases} v(p) + O(h^k) k \geq 4 & \text{if } \Delta v = 0 \\ v(p) + O(h^2) & \text{if } \Delta v \neq 0 \end{cases}$$

Assumes # interpolation points increases as h decreases.

If $\#$ interp. pts. constant as h decreases method is still $O(h^4)$ accurate near source particles.

Near a source $O(h^4)$ term in truncation error is largest-coefficient depends on high order deriv. of field.

Further from source $\leftrightarrow$ h smaller, largest term is $O(h^2)$ or $O(h)$ with very small coefficient.

Far from source, field is small

As h is halved and $\#$ mesh pts. constant $\Longrightarrow$
initial errors decrease by a factor of 16, eventually decrease by a factor of 2

Want reasonable level of accuracy near source with a coarse mesh so less of calculation done directly - these wts. provide that.

If $\Delta v \neq 0$, $I(v)$ is $O(h^2)$ accurate even when $\#$ interpolation pts. constant as h decreases-
leading term in error = error in evaluating volume integral

We can interpolate solutions of

$$\nabla \cdot \epsilon(x,y)\nabla v(x,y) = 0$$

if ϵ is piecewise constant and lines of discontinuity are straignt lines.

If $\epsilon = \epsilon_1$ for $y \le 0$, $\epsilon = \epsilon_2$ for $y > 0 \Longrightarrow$

$$G(p,q) = \begin{cases} \frac{q_2}{2\pi\epsilon_2} \log r & y \ge 0 \\ \frac{1}{2\pi\epsilon_1} (\log r + q_1 \log \bar{r}) & y < 0 \end{cases}$$

where

$$q_1 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}, q_2 = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}, \bar{r} = \sqrt{(x + x_q)^2 + (y + y_q)^2}$$

$$I(v) \approx \int_{C(R)} \epsilon(p) \left[G(p,q) \frac{\partial v(p)}{\partial n} - v(p) \frac{\partial G}{\partial n} \right] ds = v(q),$$

5 point stencil

$v(\tilde{x},\tilde{y}) = \frac{\tilde{x}}{\tilde{x}^2+\tilde{y}^2}, \tilde{x} = q_1 - x_s, \tilde{y} = q_2 - y_s,$
$|q_1 - x_s| = .04, |q_2 - y_s| = .03$ (q_1, q_2) is $(.55h, .45h)$
from lower left mesh pt.

$$\text{PMC} = \frac{\sum W(p,q)v(p)}{\sum W(p,q)}$$

Table 1

nx	PMC	Lag	CIC	PES	TSC	CA
32	.20E-1	.23E+0	.47E-1	.17E+0	.43E-1	.47E-1
64	.13E-2	.14E-1	.31E-2	.72E-3	.28E-2	.29E-2
128	.95E-4	.83E-3	.21E-3	.60E-3	.19E-3	.18E-3
256	.66E-5	.52E-4	.16E-4	.36E-3	.12E-4	.12E-4
@512	.72E-6	.32E-5	.13E-5	.18E-3	.87E-6	.72E-6
1024	.21E-6	.20E-6	.11E-6	.91E-4	.66E-7	.45E-7
2048	.97E-7	.13E-7	.12E-7	.45E-4	.56E-8	.28E-8
4096	.49E-7	.79E-9	.13E-8	.23E-4	.53E-9	.18E-9
8192	.25E-7	.49E-10	.15E-9	.11E-4	.56E-10	.11E-10

2 interpolation rings	
nx	rate conv.
32	0.000E+00
64	0.153E+02
128	0.140E+02
256	0.145E+02
512	0.909E+01
1024	0.345E+01
2048	0.215E+01
4096	0.199E+01
8192	0.199E+01

8 interpolation rings	
64	0.153E+02
128	0.143E+02
256	0.152E+02
512	0.159E+02
1024	0.160E+02
2048	0.159E+02
4096	0.150E+02
8192	0.105E+02

$v(\tilde{x}, \tilde{y}) = \tilde{x}^4 - 6\tilde{x}^2\tilde{y}^2 + \tilde{y}^4$
$|q_1 - x_s| = .012, |q_2 - y_s| = .013$
(q_1, q_2) is $(.55h, .45h)$ from lower left mesh pt.

Table 2

nx	PMC	Lag	CIC	PES	TSC	CA
32	.13E+1	.11E+2	0.24E+1	.18E+1	.25E+1	.25E-
64	.78E-1	.69E+0	0.15E+0	.11E+0	.16E+0	.15E-
128	.49E-2	.43E-1	0.97E-2	.53E-2	.11E-1	.96E-
256	.30E-3	.27E-2	0.64E-3	.42E-3	.86E-3	.60E-
512	.18E-4	.17E-3	0.44E-4	.39E-3	.72E-4	.38E-
@1024	.11E-5	.10E-4	0.32E-5	.21E-3	.68E-5	.24E-
2048	.17E-6	.65E-6	0.26E-6	.10E-3	.71E-6	.15E-
4096	.84E-7	.41E-7	0.24E-7	.52E-4	.80E-7	.92E-

2 interpolation rings		
nx	err	rate conv.
64	0.78E-01	0.16E+02
128	0.49E-02	0.16E+02
256	0.30E-03	0.16E+02
512	0.18E-04	0.17E+02
1024	0.11E-05	0.16E+02
2048	0.17E-06	0.67E+01
4096	0.84E-07	0.20E+01

8 interpolation rings		
64	0.82E-01	0.16E+02
128	0.51E-02	0.16E+02
256	0.32E-03	0.16E+02
512	0.20E-04	0.16E+02
1024	0.12E-05	0.16E+02
2048	0.78E-07	0.16E+02
4096	0.49E-08	0.16E+02

$v(\tilde{x}, \tilde{y}) = \tilde{x}^4$
$|q_1 - x_s| = .012, |q_2 - y_s| .013$
(q_1, q_2) is $(.55h, .45h)$ from lower left mesh pt.

Table 3						
nx	PMC	Lag	CIC	PES	TSC	CA
32	.38E-6	.53E-06	.31E-6	.12E-5	.31E-6	0.31E-6
64	.69E-7	.33E-07	.61E-7	.16E-6	.61E-7	0.64E-7
128	.15E-7	.21E-08	.14E-7	.31E-7	.14E-7	0.16E-7
256	.37E-8	.13E-09	.33E-8	.71E-8	.34E-8	0.43E-8
512	.91E-9	.81E-11	.83E-9	.17E-8	.83E-9	0.13E-8
1024	.23E-9	.51E-12	.21E-9	.41E-9	.21E-9	0.46E-9

2 interpolation rings		
64	0.69E-07	0.55E+01
128	0.15E-07	0.45E+01
256	0.37E-08	0.41E+01
512	0.91E-09	0.41E+01
1024	0.23E-09	0.40E+01
2048	0.57E-10	0.40E+01
4096	0.14E-10	0.40E+01

$v(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{x}{(x^2+y^2+z^2)^{1.5}}$
$|(q_1, q_2, q_3)) - (x_s, y_s, z_s)| = (.06, .04, .08)$
(q_1, q_2, q_3) is $(.45h, .40h, .52h)$ from lower left mesh pt.

Table 4				
nx	PMC	CIC	TSC	conv. rate
33	0.15E-2	0.22E-2	0.23E-02	0.160E+02
65	0.56E-4	0.25E-3	0.34E-03	0.263E+02
129	0.55E-5	0.12E-4	0.48E-04	0.102E+02
257	0.16E-5	0.41E-5	0.63E-05	0.340E+01
@513	0.75E-6	0.20E-5	0.81E-06	0.214E+01
1025	0.37E-6	0.62E-6	0.10E-06	0.202E+01
2049	0.19E-6	0.17E-6	0.13E-07	0.201E+01
4097	0.93E-7	0.44E-7	0.16E-08	0.201E+01
8193	0.46E-7	0.11E-7	0.20E-09	0.200E+01

$v(\tilde{x}, \tilde{y}, \tilde{z} = \tilde{x}^4 + \tilde{y}^4 + \tilde{z}^4$
$|(q_1, q_2, q_3)) - (x_s, y_s, z_s)| = (.06, .04, .08)$
(q_1, q_2, q_3) is $(.45h, .40h, .52h)$ from lower left mesh pt.

Table 5

nx	PMC	CIC	TSC	conv. rate
17	.73E+0	.60E+00	.63E+00	.
33	.18E+0	.15E+00	.16E+00	0.401E+01
65	.45E-1	.38E-01	.39E-01	0.400E+01
129	.11E-1	.94E-02	.98E-02	0.400E+01
257	.28E-2	.24E-02	.24E-02	0.401E+01
513	.70E-3	.59E-03	.61E-03	0.401E+01
1025	.17E-3	.15E-03	.15E-03	0.403E+01
2049	.43E-4	.37E-04	.38E-04	0.406E+01
4097	.10E-4	.92E-05	.96E-05	0.412E+01
8193	.24E-5	.23E-05	.24E-05	0.426E+01

$\nabla\epsilon\nabla u = 0$, $\epsilon = \epsilon_1 = 21, x < 0$, $\epsilon = \epsilon_2 = 1, x \geq 0$
$v(q) = x$ component of field due to particle at distance (.195,,055) from interpolation pt., both in R_1
(q_1, q_2) is $(.45h, .75h)$ from lower left mesh pt.

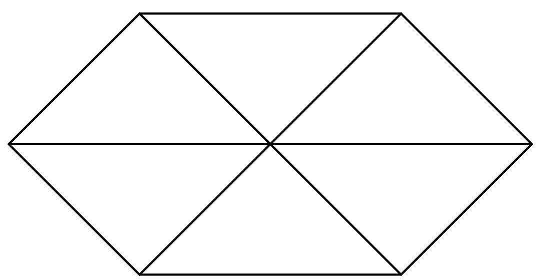
Table 6					
nx	PMC	Lag	CIC	PES	TSC
16	.21E-6	.87E-4	.34E-5	.16E-3	.12E-5
32	.61E-7	.42E-4	.12E-5	.70E-4	.10E-6
64	.59E-8	.21E-4	.37E-6	.32E-4	.10E-7
128	.51E-9	.98E-5	.88E-7	.16E-4	.10E-8
256	.76E-10	.47E-5	.22E-7	.76E-5	.10E-9
512	.26E-10	.23E-5	.47E-8	.38E-5	.11E-10

$$\nabla \cdot \left(\frac{1}{1 + k^2 r^2} \nabla u \right) u = 0$$

$k =$ constant

$$G(r) = \frac{1}{2\pi} \left(\log r + \frac{k^2 r^2}{2} \right)$$

Form $\{W^i\}$ using 7 point discretization on triangular grid - $O(h^4)$ accurate



Interpolant $\approx$ trapezoid rule approx. to integral in Green's formula

C periodic $\Longrightarrow$ trapezoid rule accurate.

$v(\tilde{x},\tilde{y}) = \frac{\tilde{x}}{\tilde{x}^2+\tilde{y}^2} + k^2\tilde{x}, k = 10$
$|q_1 - x_s| = .04, |q_2 - y_s| = .035$ (q_1, q_2) is $(.53h, .47h)$ from lower left mesh pt.

2 interpolation rings		
nx	rel. error	conv. rate
32	0.61E+00	0.00E+00
64	0.25E+00	0.25E+01
128	0.28E-02	0.90E+02
256	0.62E-03	0.45E+01
512	0.42E-04	0.15E+02
1024	0.26E-05	0.16E+02
2048	0.11E-06	0.25E+02

$\Delta u - k^2 u = 0, 3D$

$G(r) = \frac{e^{-kr}}{r}$

$W^i = \left(\Delta_h - k^2 I\right) G$

$v(\tilde{x}, \tilde{y}, \tilde{z}) = \left(\frac{x}{(r)^{1.5}} - \frac{xk}{r^2}\right) e^{-kr}, k = 1.5$
$|(q_1, q_2, q_3)) - (x_s, y_s, z_s)| = (.05, .04, .05)$
(q_1, q_2, q_3) is $(.4h, .25h, .32h)$ from lower left mesh pt.

Table 7

nx	PMC	CIC	TSC	rate conv
32	0.115E+01	0.889E+00	0.467E+00	0.000E+00
64	0.610E-01	0.310E+00	0.681E-01	0.188E+02
128	0.656E-02	0.623E-01	0.118E-01	0.929E+01
@256	0.146E-02	0.129E-01	0.147E-02	0.450E+01
512	0.506E-03	0.283E-02	0.146E-03	0.288E+01
1024	0.210E-03	0.658E-03	0.653E-05	0.241E+01
2048	0.951E-04	0.158E-03	0.225E-05	0.221E+01

Stokes Equations

$$\Delta U(X) - \nabla p(X) = F(X)$$
$$\nabla . U(X) = 0$$

Fundamental solutions in $2D$:
e_j unit vector in the j direction

$$\Delta U(X) + \nabla p(X) = e_j \delta(X - Y)$$
$$\nabla . U(X) = 0$$
$$U_{i,j}(X - Y) = \frac{1}{4\pi}\left[\delta_{i,j} \log \frac{1}{|X-Y|} + \frac{(X_i - Y_i)(X_j - Y_j)}{|X-Y|^2}\right]$$
$$p_j(X,Y) = \frac{1}{2\pi}\frac{\partial}{\partial x_j} \log \frac{1}{|X-Y|}$$
$$U_{i,j}(X - Y) = \frac{-1}{8\pi}\left[\frac{\delta_{i,j}}{|X-Y|} + \frac{(X_i - Y_i)(X_j - Y_j)}{|X-Y|^3}\right]$$
$$p_j(X,Y) = -\frac{X_i - Y_i}{4\pi|X-Y|^3}$$

in 3D.

Solution at $X = (x, y)$ due to unit force in the x direction at $q = (q_1, q_2)$:

$$u_1(x, y) = -\frac{1}{4\pi} \log |X - q| + \frac{(x - q_1)^2}{(x - q_1)^2 + (y - q_2^2)}$$

$$v_1(x, y) = \frac{1}{4\pi} \frac{(x - q_1)(y - q_2)}{(x - q_1)^2 + (y - q_2)^2)},$$

$$p_1(x, y) = \frac{1}{2\pi} \frac{\partial}{\partial x} \log \frac{1}{|X - q|}$$

Symmetric equations hold for a unit force in the y direction.

L_h = discrete approximation to Stokes operator used in the solver,
U_i, p : fundamental solution.
$\Longrightarrow W^i = L_h(U_i, p)$

Δ_h and D_h^x discrete approximations used in the Stokes solver
$\Longrightarrow$ force distributed to mesh point due to unit force in x direction

$$\Delta_h u_1 - D_h^x p_1$$

Same weights used for interpolating x component velocity onto the the particles

Velocity components in the direction of force have same singular parts as solutions of Laplace's equation.

$\Delta\frac{(x-q_1)^2}{(x-q_1)^2+(y-q_2^2)} - \frac{\partial^2}{\partial x^2}\log\frac{1}{|X-q|} = 0 \Longrightarrow$
other parts of the weights are small.

$\Longrightarrow$ set $W^i = 0$ except near source.

Solving for velocities different from solving for a potential:

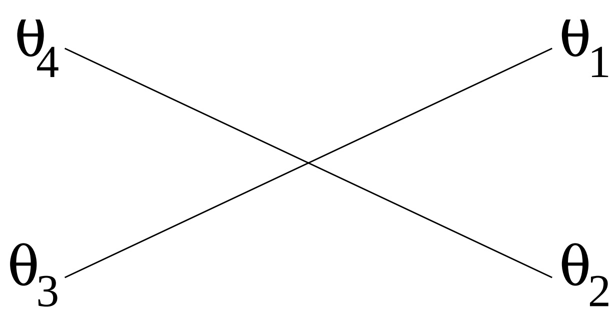
In the abscence of other forces maximum velocity due to force is where the force is applied.

Velocity at p due to force at q equal
in magntitude and opposite in direction to
velocity at q due to opposite force at p.

$$\sum W^i = \frac{1}{4\pi}\left(\sum \Delta_h \log\left(\frac{1}{r}\right) + \sum \Delta_h \frac{x^2}{r^2} + \sum D_h^x \log\left(\frac{1}{r}\right)\right)$$

$$= I + II + III$$

$I = \frac{1}{2}$
$II = \frac{1}{4\pi}\left(\theta_1 - \theta_2 + \theta_3 - \theta_4 + ...\right)$ depends on shape of interpolation region
$III = \frac{1}{4\pi}\left(\theta_1 - \theta_4 + \theta_3 - \theta_2 + ...\right)$
$II + III \approx \frac{1}{2}$



$\Longrightarrow \sum W^i \approx 1$

Weights are $O(h^2)$ accurate for interpolation

$v(\tilde{x}, \tilde{y}) = \log r - \frac{\tilde{x}^2}{r^2}$,
$|q_1 - x_s| = .048, |q_2 - y_s| = .74$
(q_1, q_2) is $(.45h, .55h)$ from lower left mesh pt.

Table 8 2D Stokes velocity		
nx	error	rate conv.
32	0.766E-04	0.000E+00
64	0.191E-04	0.400E+01
128	0.478E-05	0.401E+01
256	0.119E-05	0.402E+01
512	0.295E-06	0.404E+01
1024	0.723E-07	0.407E+01
2048	0.174E-07	0.415E+01
4096	0.402E-08	0.433E+01
8192	0.838E-09	0.480E+01

$v(\tilde{x},\tilde{y}) = \tilde{x^3}$,
$|q_1 - x_s| = .048, |q_2 - y_s| = .74$
(q_1, q_2) is $(.45h, .55h)$ from lower left mesh pt.

Table 9 2D Stokes		
nx	error	rate conv.
32	0.268E-03	0.000E+00
64	0.669E-04	0.400E+01
128	0.167E-04	0.400E+01
256	0.418E-05	0.400E+01
512	0.104E-05	0.400E+01
1024	0.260E-06	0.401E+01
2048	0.648E-07	0.402E+01
4096	0.161E-07	0.403E+01
8192	0.394E-08	0.407E+01

$v(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{2}{r} + \frac{x^2}{r^3}$
$|(q_1, q_2, q_3)) - (x_s, y_s, z_s)| = (.06, .04, .68)$
(q_1, q_2, q_3) is $(.54h, .25h, .5h)$ from lower left mesh pt.

Table 10 3D Stokes velocity		
nx	error	rate conv.
16	0.104E-02	0.000E+00
32	0.259E-03	0.403E+01
64	0.645E-04	0.402E+01
128	0.160E-04	0.402E+01
256	0.397E-05	0.404E+01
512	0.977E-06	0.407E+01
1024	0.236E-06	0.414E+01
2048	0.547E-07	0.431E+01
4096	0.116E-07	0.473E+01
8192	0.183E-08	0.631E+01

$v(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{x}^4 + \tilde{y}^4 + \tilde{z}^4$
$|(q_1, q_2, q_3)) - (x_s, y_s, z_s)| = (.06, .04, .68)$
(q_1, q_2, q_3) is $(.54h, .25h, .5h)$ from lower left mesh pt.

Table 11 3D Stokes		
nx	error	rate conv.
16	-0.131E-01	0.000E+00
32	-0.328E-02	0.399E+01
64	-0.820E-03	0.400E+01
128	-0.205E-03	0.399E+01
256	-0.514E-04	0.399E+01
512	-0.129E-04	0.398E+01
1024	-0.326E-05	0.397E+01
2048	-0.827E-06	0.394E+01
4096	-0.213E-06	0.388E+01
8192	-0.567E-07	0.377E+01

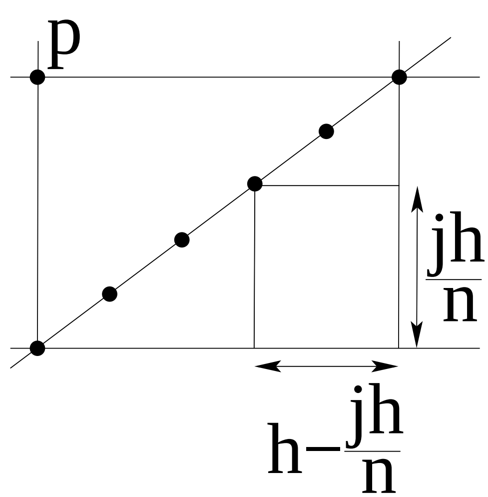
Continuous distribution of particles on curve or surface

$F = \log r$ or $\frac{1}{r} \Longrightarrow$ field is field induced by vortex sheet:

$$V = \frac{1}{2\pi} \int_C \mu \log r ds$$

Field can often be evaluated analytically

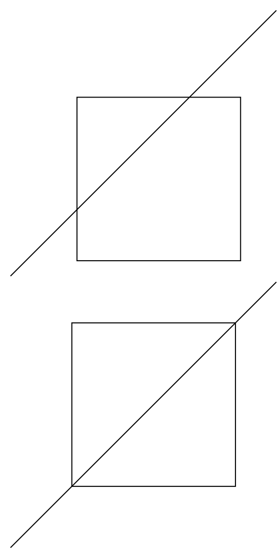
Can also often evaluate results of standard PM methods



Add contribution of each particle and take limit

Results of P.M. methods smoother, but more mesh dependent

Some methods $O(h^2)$ if C parallel to coordinate axis, $O(h)$ accurate otherwise



Analytic solution 2D stationary Stokes problem on region with force on curve in fluid

$$\mu \Delta U - \nabla p = F$$
$$\nabla U = 0$$

F only nonzero on $D(s)$

$$\mu \Delta u - p_x = \int_{D(s)} f_1(s) \delta(\bar{x} - X(s)) ds$$

$$\mu \Delta v - p_y = \int_{D(s)} f_2(s) \delta(\bar{x} - X(s)) ds$$
$$u_x + v_y$$

$p =$ pressure, $(u, v) =$ velocity, $\mu =$ viscosity

Formulation of solution of Stokes problem in terms of Cauchy integrals, the Goursat functions

Except on interface

$$\mu\Delta u = p_x$$

$$\mu\Delta v = p_y$$

$$u_x + v_y = 0$$

$$\Longrightarrow \Delta^2 W = 0$$

$$W = \text{Re}\,\left(\overline{z}\phi(z) + \chi(z)\right)$$

$\phi(z), \psi(z) = \chi'(z)$ Goursat functions

$$u = W_y,\ \ v = -W_x$$

$$-v = W_x = x\phi_{1x} + \phi_1 + y\phi_{2x} + \psi_1$$

$$u = W_y = x\phi_{1y} + \phi_2 + y\phi_{2y} - \psi_2$$

ϕ, ψ complex analytic

$\Longrightarrow$ they can be expressed as Cauchy integrals

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\omega(t)}{t - z} dt; \;\; \psi(z) = \frac{1}{2\pi i} \int \frac{\bar{\omega}(t) - t' \omega'(t)}{t - z} dt$$

If we can determine the densities ω
$\Longrightarrow$ the problem reduces to evaluating Cauchy integrals, even though the velocity is not harmonic and p not known a priori.

Can be done numerically using 4 independent Poisson solvers

or (sometimes) analytically

To determine ω

Shape and force on boundary determine discontinuities in stress tensor across $D(s)$

These discont. and continuity of u, v in tan. dir. $\Longrightarrow$ discont. in deriv. of u, v and p.

Discont. in Cauchy integrals determined by their density functions

$\Longrightarrow$ Density function can be determined

Once their densities are determined, the integrals can be evaluated

$$\omega_1'(s) = -\frac{f_1(s)d(s)}{4} \text{ and } \omega_2'(s) = -\frac{f_2(s)d(s)}{4}$$

$$d(s) = \sqrt{(x'(s)^2 + (y'(s))^2}$$

Whenever the Cauchy integrals can be evaluated analytically, so can the velocity and pressure.

This can often be done when $C(s)$ is a circle or ellipse