Blob Methods for Free Surface Flows

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Features of Blob Methods

- Blobs are a regularization of free surface motion if the it is ill-posed. Best example is a vortex sheet, an interface between two immiscible liquids of the same density.

- Blobs provide a simple way to construct accurate approximations to boundary integrals. The application is important in three-dimensions.

Boundary Integrals for Free Surface Flows

Assume the surface has the parametric form $\mathbf{x}(p,q)$ and there is a vortex sheet on the surface, $\boldsymbol{\gamma} = \gamma(p,q)\,\mathbf{t}$, where $\mathbf{t}$ is a unit tangent vector. The velocity induced by the vortex sheet is

$$\mathbf{u}(\mathbf{x},t) = \int \gamma(p,q)\,\mathbf{t}(p,q) \times \nabla G\Big(\mathbf{x} - \mathbf{x}(p,q)\Big)\,\mathrm{d}S(p,q)\,,$$

where $G(\mathbf{x})$ is the free surface Greens function for Laplace's equation.

When x lies on the surface, the principal-values of the integral is taken.

The vortex sheet strength measures the jump in tangential velocity components; Normal components continuous.

$$\gamma(p,q)\,\mathbf{n}(p,q) \times \mathbf{t}(p,q) = \mathbf{u}_+(p,q) - \mathbf{u}_-(p,q)$$

Boundary Integrals for Free Surface Flows Continued

Evolution equation for γ based on dynamic considerations.

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \nabla p \times \nabla \left(\frac{1}{\rho}\right)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho}$$

Evaluation at the surface leads to boundary integral equations - Fredholm integral equations of the second kind.

$$\nabla p \approx \frac{\partial \mathbf{u}}{\partial t} \approx \int \frac{\partial \boldsymbol{\omega}}{\partial t} K\,, \quad \rho \nabla \left(\frac{1}{\rho}\right) \approx \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

Details available elsewhere. Note, if $\nabla \rho = 0$, classic vortex sheet motion.

Specific Surface Integrals

In three-dimensions,

$$\mathbf{u}(p,q) = \frac{1}{4\pi}\int \gamma(\alpha,\beta)\,\mathbf{t}(\alpha,\beta) \times \frac{\mathbf{x}(p,q)-\mathbf{x}(\alpha,\beta)}{|\mathbf{x}(p,q)-\mathbf{x}(\alpha,\beta)|^3}\,\mathrm{d}S(\alpha,\beta)$$

In two-dimensions, $\mathbf{t}$ now points in third direction.

$$\mathbf{u}(p) = \frac{1}{2\pi}\int \gamma(\alpha)\,\mathbf{t} \times \frac{\mathbf{x}(p)-\mathbf{x}(\alpha)}{|\mathbf{x}(p)-\mathbf{x}(\alpha)|^2}\,\mathrm{d}S(\alpha)$$

The singularity in the integrand can be weakened by using appropriate versions of

$$\int \mathbf{n}(\alpha,\beta) \times \nabla G(\mathbf{x}(p,q)-\mathbf{x}(\alpha,\beta)\,\mathrm{d}S(\alpha,\beta) = 0$$

In two-dimensions, the singularity is removed completely and standard quadratures methods (trapezoidal rule) are spectrally accurate. In three-dimensions, a square root remains in the denominator.

Blob Methods

Regularize the integrand.

$$\mathbf{u}_\delta(p,q) = \int \mathbf{t}(\alpha,\beta) \times \nabla G\Big(\mathbf{x}(p,q) - \mathbf{x}(\alpha,\beta)\Big)\, H\Big(r/\delta)\Big)\, \mathrm{d}S(\alpha,\beta)\,,$$

where $r = |\mathbf{x}(\alpha,\beta) - (x)(p,q)|$. Require $H(r)/r$ to be smooth function of r^2 and $H(r) \to 1$ quickly as $r \to \infty$. The error incurred is

$$E_\delta = \int \mathbf{F}(\alpha,\beta) \left(H\Big(r/\delta)\Big) - 1\right) \mathrm{d}S(\alpha,\beta)\,,$$

where

$$\mathbf{F}(\alpha,\beta) = \mathbf{t}(\alpha,\beta) \times \nabla G\Big(\mathbf{x}(p,q) - \mathbf{x}(\alpha,\beta)\Big)$$

Specific Choices for H

Introduce polar coordinates centered on $\mathbf{x}(p,q)$, and expand. Typical contribution to the error is

$$I_n = \int_0^\infty r^n \left[H\left(\frac{r}{\delta}\right) - 1 \right] \mathrm{d}r$$

$I_n = 0$ for n odd. If $I_0 = 0$, then dominant error is $I_3 \approx c\,\delta^3$.

Beale (Majda, Hald, Goodman) gives some explicit examples.

$$H(r) = \mathrm{erf}(r) + \frac{2}{\sqrt{\pi}} (2r^3 - r)\, \mathrm{e}^{-r^2} \qquad \text{3D}$$

$$H(r) = 1 + (2r^2 - 1)\, \mathrm{e}^{-r^2} \qquad \text{2D}$$

Trapezoidal approximation to the integral for $\mathbf{u}_\delta$ is $\mathrm{O}(h^3)$. Balance δ with$h$. Correction may be needed at singular point.

Test Cases in 2D

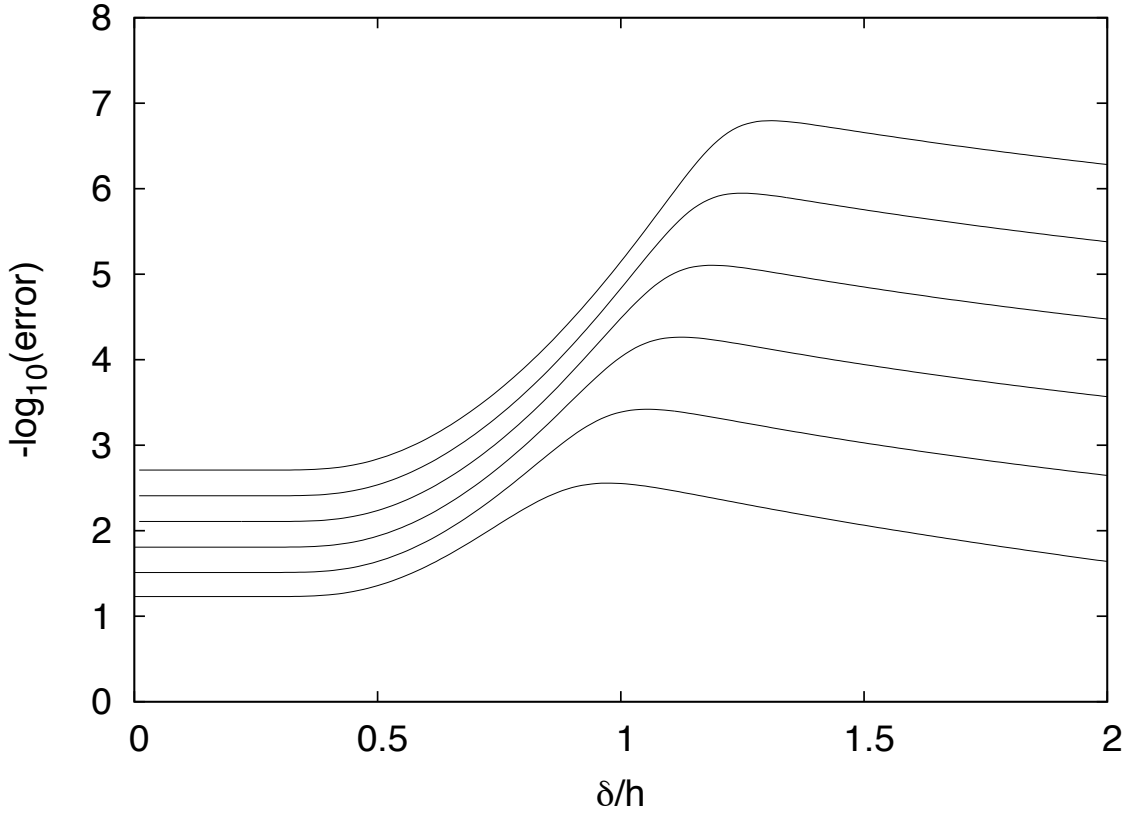
Pick a family of ellipses.

$$x(p) = \cos(p)\,, \quad y(p) = b\sin(p).$$

Use conformal map to solve the Laplace's equation. Match solution inside and outside so that the normal velocity is continuous. Determine γ and $\mathbf{u}$

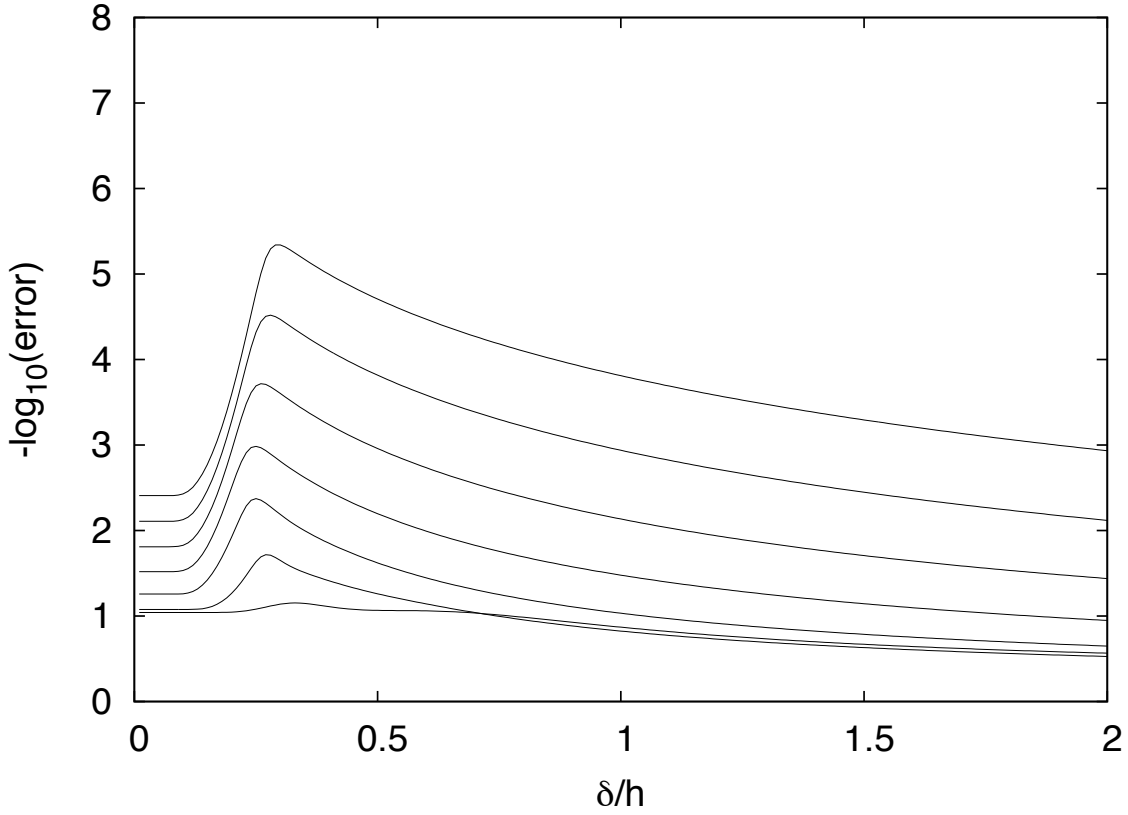
Use N points, and vary δ. Results are shown as max abs errors in $\mathbf{u}$ for a circle and a 4-to-1 ellipse.

Results for Circle



Errors for $N = 16, 32, 64, 128, 256$ from the bottom.

Results for 4-to-1 Ellipse



Errors for $N = 16 \rightarrow 1024$.

Rescaled δ

The shift in location in δ/h is associated with the variation in arclength spacing. Where the error is largest, $\triangle s \approx h/4$. Thus $\delta/\triangle s \approx 1.2$, consistent with results from the circle.

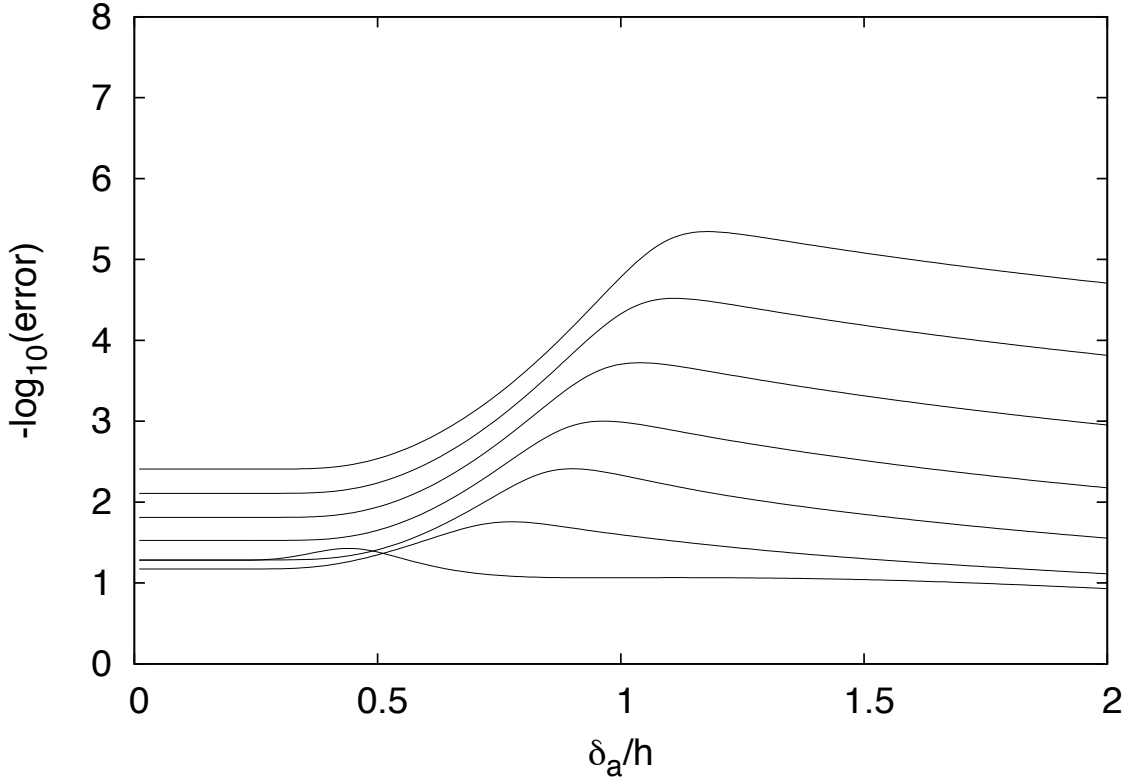
Replace

$$H\left(\frac{|\mathbf{x}(\alpha) - \mathbf{x}(p)|}{\delta}\right) \quad \text{by} \quad H\left(\frac{|\mathbf{x}(\alpha) - \mathbf{x}(p)|}{s_p(p)\delta_a}\right)$$

where

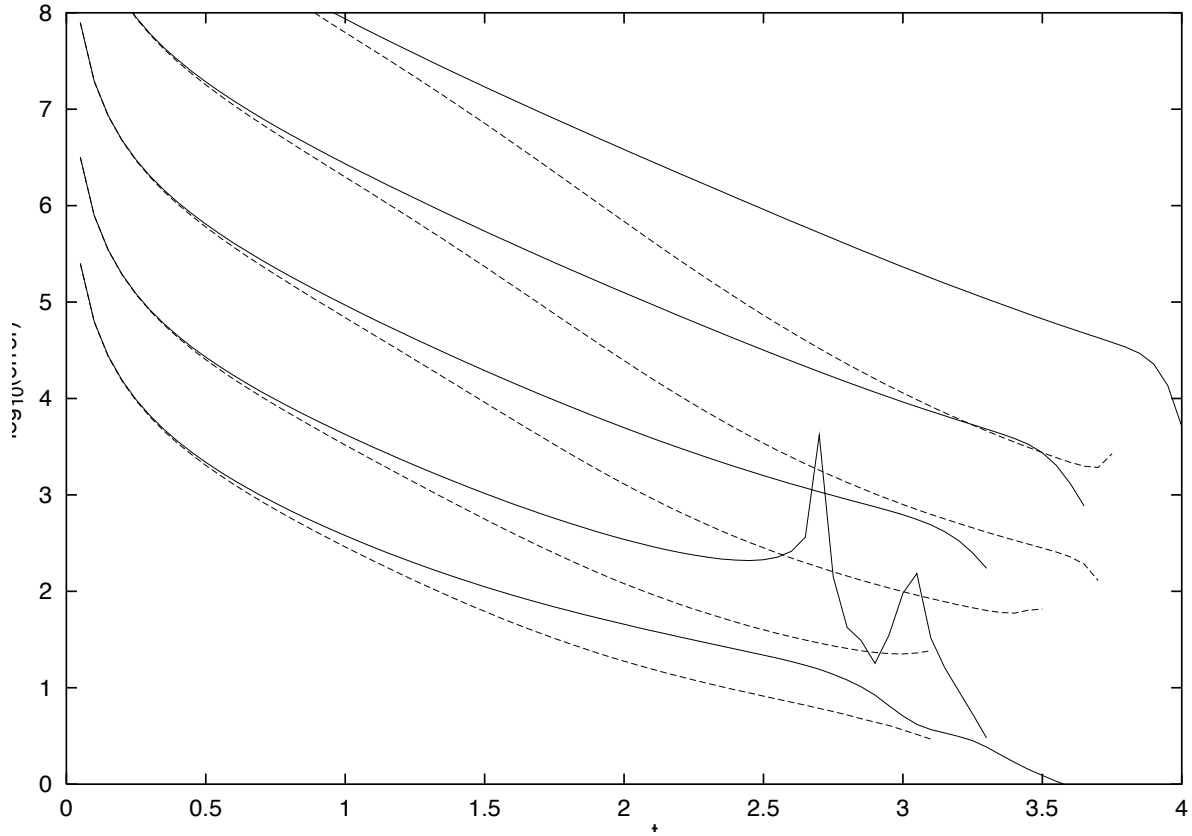
$$s_p^2 = x_p^2 + y_p^2$$

Results for Rescaled δ



Errors for $N = 16 \rightarrow 1024$.

Application to Classical Rayleigh-Taylor Instability



Errors in time for $N = 32 \rightarrow 512$: solid $\delta = 2s_p h$; dashed $\delta = 2h$.

Test Cases for 3D

Pick a cylinder,

$$x(p,q) = \cos(p)\,, \quad y(p,q) = \sin(p)\,, \quad z(p,q) = q$$

Solve Laplace's equation with potentials of the form

$$\phi = f(r)\,\cos(n\theta)\,\cos(az)$$

where $f(r)$ satisfies the modified Bessel equation

$$f(r) \sim A\,\frac{I_n(ar)}{aI_n'(ar)} \quad \text{or} \quad f(r) \sim B\,\frac{K_n(ar)}{aK_n'(ar)}$$

Leads to

$$\boldsymbol{\gamma} = n\sin(np)\,\cos(aq)\hat{\mathbf{z}} - a\,\cos(np)\,\sin(aq)\mathbf{t}(p)$$

where $\mathbf{t}(p)$ is the unit tangent to the circular cross-section.

Periodicity Along the Cylinder

Replace infinite integration by finite range through the method of images.

$$\int_{-\infty}^{\infty} F(q)\,\mathrm{d}q = \int_{0}^{2\pi/a} \sum_{k=-\infty}^{\infty} F(q + 2k\pi/a)\,\mathrm{d}q\,.$$

Apply to

$$\sum_{k=-\infty}^{\infty} \frac{H(r_k/\delta)}{r_k^3} \quad \text{and} \quad \sum_{-\infty}^{\infty} (\eta - q - 2k\pi/a)\,\frac{H(r_k/\delta)}{r_k^3}\,,$$

where $r_k^2 = (x(p) - x(\alpha))^2 + (y)p) - y(\alpha))^2 + (q - \beta + 2k\pi/a)^2$.

Use symmetry and far-field decay, for example, first sum becomes

$$\frac{1}{2} \sum_{k=-\infty}^{\infty} \left[\frac{H(r_k/\delta)}{r_k^3} + \frac{H(r_{-k}/\delta)}{r_{-k}^3} - \frac{\alpha^3}{4\pi^3|k|^3} \right] + \frac{a^3}{4\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3}\,.$$

Results for $n = 1$ and $a = 1$

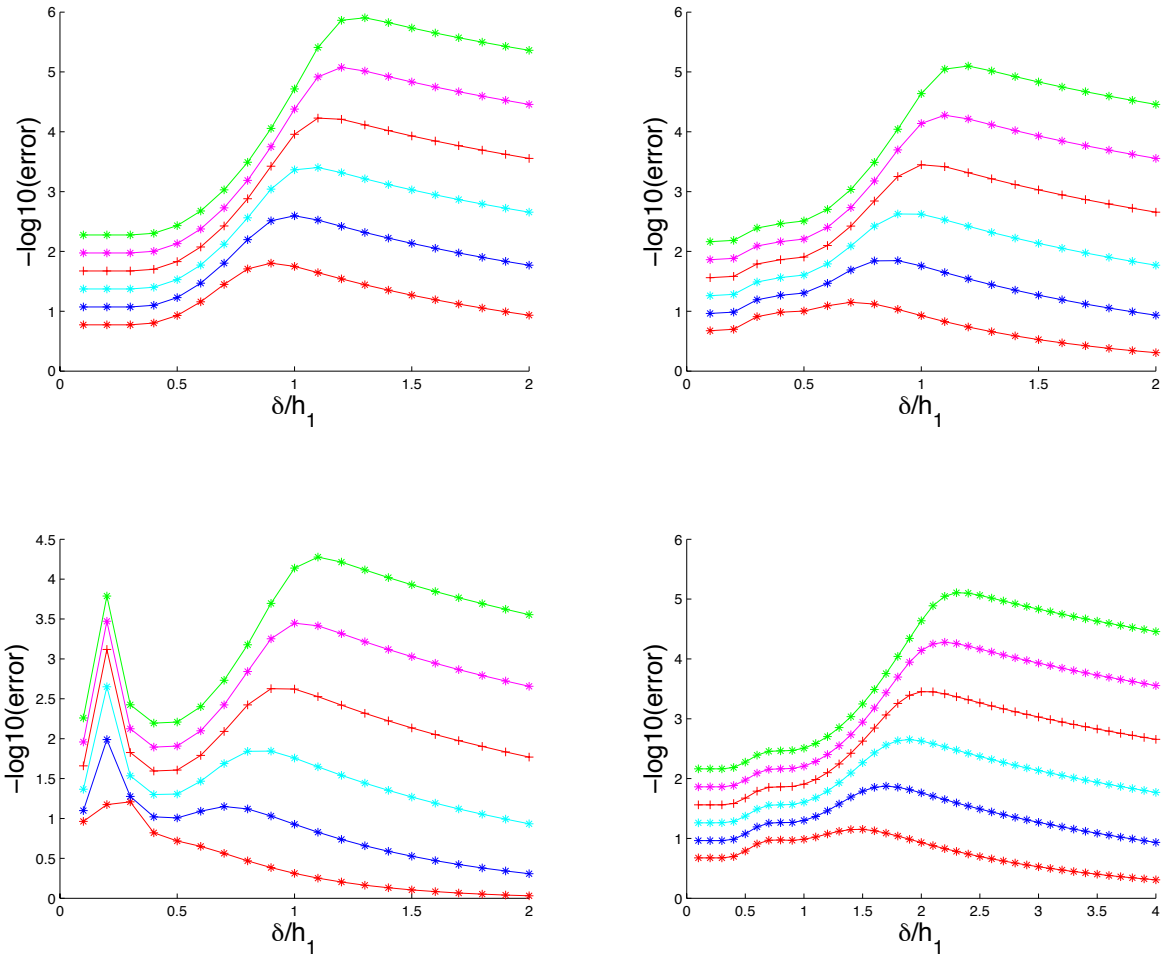
Let N and M be number of points in p and q. Then

$$h_1 = 2\pi/N \quad \text{and} \quad h_2 = 2\pi/(aM)$$

Explore the role of δ/h_1 and δ/h_2 by fixing N and M and so the ratio h_1/h_1 is fixed and vary δ.

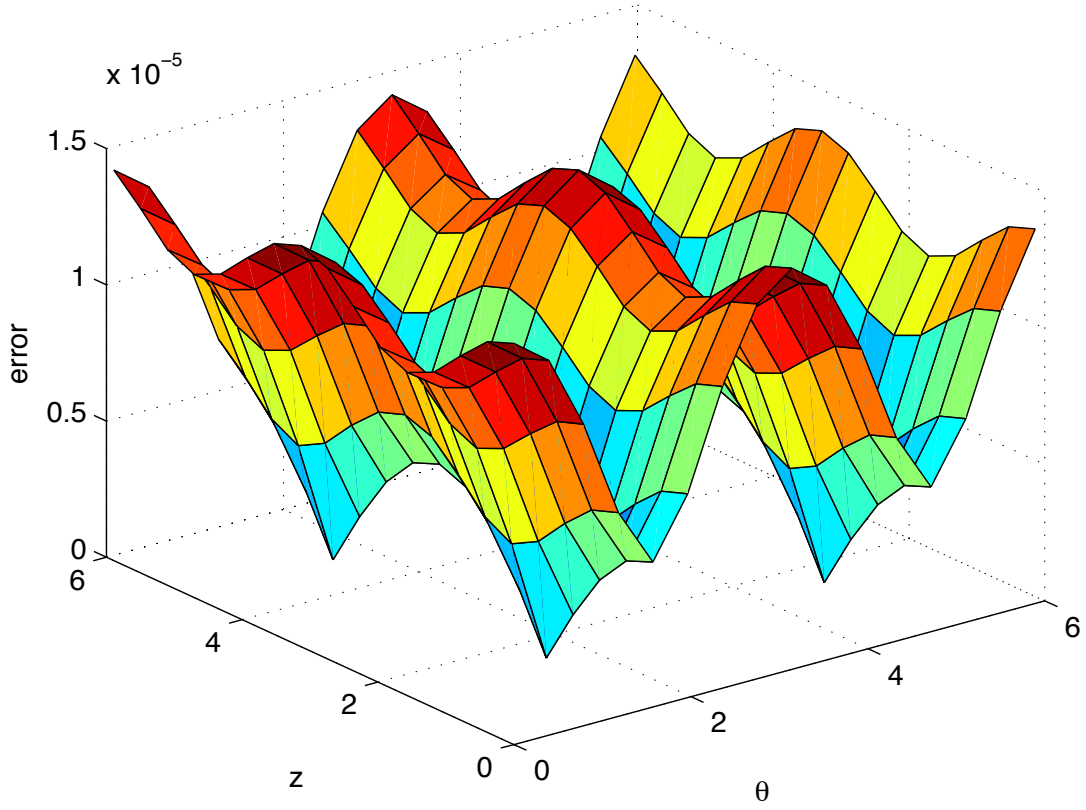
Repeat by increasing N and M in the same ratio.

Note in this simple example, physical spacing is the same as h_1 and h_2.



Top/left; $h_1/h_2 = 1$, Top /right; $h_1/h_2 = 2$. Bottom/left; $h_1/h_2 = 4$. Bottom/right; $h_1/h_2 = 0.5$.

Smoothness of the Errors



Max abs error for $\delta/h_1 = \delta/h_2 = 1.5$ and $N = 256$.

Preliminary Conclusions

- The grid spacing must be just smaller than δ - this may become problematic under stretching and deformation under motion

- The error profile is smooth - important for subsequent motion.