Global and almost global wellposedness of the two and three dimensional full water wave equations

Sijue Wu

University of Michigan, Ann Arbor

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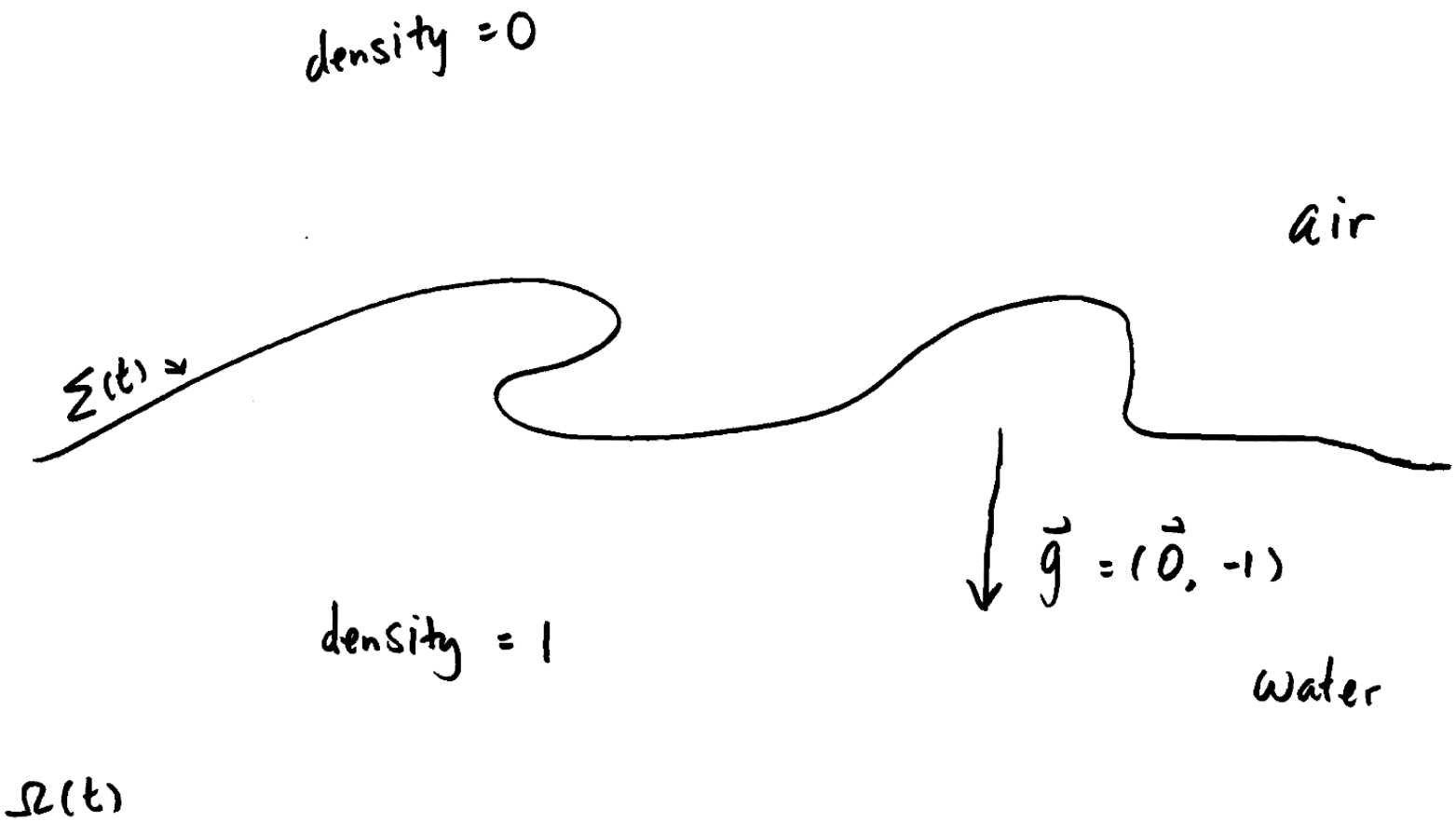
We consider the motion of the interface separating air from water of infinite depth.

We assume:

- air density= 0
- water density =1
- water region is below the air region. At time t, water region is $\Omega(t)$, the interface is $\Sigma(t)$.

We assume that the water is

- inviscid, incompressible, irrotational.
- The surface tension is zero.
- The water is subject to the influence of gravity $\mathbf{g} = (\mathbf{0}, -1)$.


density = 0
air
$\Sigma(t)$
$\vec{g} = (\vec{0}, -1)$
density = 1
water
$\Omega(t)$

The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = (\mathbf{0}, -1) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \Sigma(t) \end{cases} \tag{1}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

When surface tension is zero, the motion can be subject to the Taylor instability.

- Taylor condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$$

on the interface $\Sigma(t)$. $\mathbf{n}$ is the unit normal to $\Sigma(t)$ pointing out of the water region $\Omega(t)$.

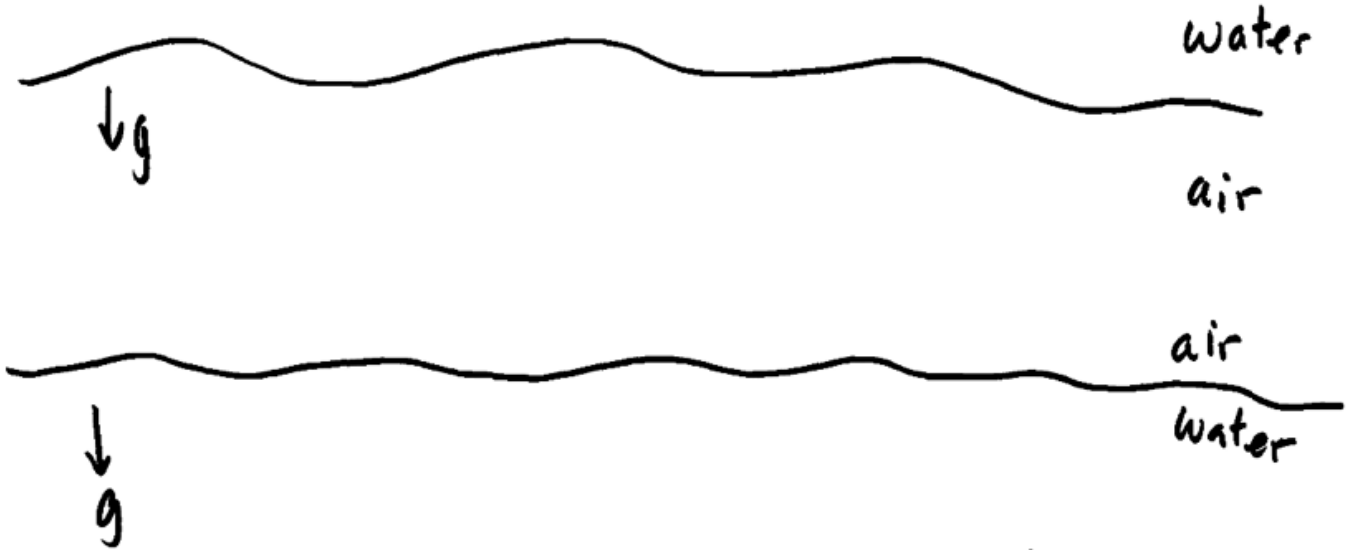
G. I. Taylor: linearize about the flat interface,

- air above water

$$-\frac{\partial P}{\partial \mathbf{n}} > 0 : \quad \text{stable.}$$

- water above air

$$-\frac{\partial P}{\partial \mathbf{n}} < 0 : \quad \text{unstable.}$$



Known results:

local wellposedness for arbitrary data

S. Wu: (1997) 2-D, (1999) 3-D.

- The Taylor condition always holds for the water wave motion, i.e.
$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$$
as long as the interface is non-selfintersecting.
- Local existence and uniqueness in Sobolev spaces: There exist a unique solution for a finite time period, for any initially non-selfintersecting interface, and any given initial velocity (incompressible & irrotational).

The proof of the fact $-\dfrac{\partial P}{\partial \mathbf{n}} > 0$:

Applying div to both sides of the Euler equation gives

$$\Delta P = -|\nabla u|^2 \leq 0$$

Maximum principle implies $-\dfrac{\partial P}{\partial \mathbf{n}} \geq 0$.

Earlier results:

1. T. Beale, T. Hou & Lowengrub (1992).
   Linear wellposedness assuming the presumed solution satisfies the Taylor sign condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0.$$

2. Nalimov(1974): infinite depth, 2-D, small data, local wellposedness
3. Yosihara(1982): finite depth, 2-D, small data. local wellposedness

Recent works:

Local wellposedness with additional effects: nonzero surface tension, finite depth, nonzero vorticity, assuming the Taylor sign condition holds.

- Iguchi(2001), Ogawa & Tani (2002), Ambrose & Masmoudi(2005), D. Lannes (2005), Christodoulou & Lindblad (2000), Lindblad (2005), Coutand & Shkoller (2007), P. Zhang & Z. Zhang (2007), Shatah & Zeng (2008)

What is the global in time behavior of the solution for the water wave equation?

We will focus on small and smooth data.

2-D water wave:

- S. Wu (2008, publication in Inventions 2009):
- For data of type $\epsilon\Psi$, there exists a unique smooth solution for time period $0 \leq t \leq e^{c/\epsilon}$.
- c only depend on Ψ.

Key idea:

- The quantity $\Theta = (I - \mathfrak{H})y$ in an appropriate coordinate system satisfies

$$\partial_t^2 \Theta - i\partial_\alpha \Theta = G \tag{2}$$

- G consisting of only cubic and higher order terms.
- y: the height function for the interface $\Sigma(t) : (x, y) = (x(\alpha, t), y(\alpha, t))$
- $\mathfrak{H}$: the Hilbert transform related to water region $\Omega(t)$.

The advantage of the structure:

Linearize the water wave equation about the flat interface:

$$\partial_t^2 u + |D|u = \text{nonlinear terms}$$

- Nonlinear term contains quadratic and higher order terms.
- The linear equation is globally well-posed.
- In $2-D$, the $L^1 \to L^\infty$ decay rate is $1/t^{1/2}$,
- In $3-D$, the $L^1 \to L^\infty$ decay rate is $1/t$.
- Nonlinear interaction can cause finite time blow up of solutions.
- Quadratic interaction is too strong.
- For small solution, the higher the order, the weaker the nonlinear interaction.

Heuristics

$$\partial_t^2 u + |D|u = u^n \partial_t u$$

(If $|u(t)|_\infty \leq \frac{E(t)^{1/2}}{(1+t)^{1/2}}$ in 2D, $|u(t)|_\infty \leq \frac{E(t)^{1/2}}{1+t}$ in 3D,)

Energy $E(t) = \int |\partial_t u|^2 + ||D|^{1/2} u|^2$ satisfies:

$$\frac{dE(t)}{dt} \leq |u|_\infty^n E(t) \leq \frac{1}{(1+t)^{n/2}} E(t)^{1+n/2} \qquad \text{2-D}$$

$$\frac{dE(t)}{dt} \leq |u|_\infty^n E(t) \leq \frac{1}{(1+t)^{n}} E(t) \qquad \text{3-D}$$

Does 3-D water wave posses such special structure?

A natural setting for studying 3-D water wave is Clifford Algebra and use Clifford analysis.

Difficulties in 3-D

- There is no Riemann mapping in 3-D.
- Clifford Algebra is non commutative.
- products of analytic functions in 3-D are not analytic.

Answer: We find in 3-D water wave there is also such special structure.

3-D water wave:

- The quantity $\theta = (I - \mathfrak{H})z$ satisfies such equation

$$\partial_t^2 \theta - \mathfrak{a} N \times \nabla \theta = G \tag{3}$$

- G is a nonlinearity of cubic and higher orders in nature.
- There is a coordinate change k so that in this new coordinate system, $\partial_t^2 \theta - \mathfrak{a} N \times \nabla \theta$ is linear+"cubic" and higher order terms.

Here:

- $\Sigma(t) : \xi = (x(\alpha, \beta, t), y(\alpha, \beta, t), z(\alpha, \beta, t))$ is the interface in Lagrangian coordinates $(\alpha, \beta) \in R^2$,
- $\mathfrak{H}$ is the Hilbert transform associated to the water region $\Omega(t)$,
- $N = \xi_\alpha \times \xi_\beta$ is the outward normal to $\Sigma(t)$.

Statement of Theorems.

Theorem (2-D water wave)

Assume that

- *initial height function is* $y_0(x) = \epsilon f(x)$
- *initial velocity is* $\mathbf{v}_0(x, y) = \epsilon g(x, y)$

$f \in L^2(\Sigma(0))$, $g \in L^2(\Omega(0))$, *and finitely many derivatives of* f *and* g *are in* L^2.

There is $\epsilon_0 > 0$, *such that if* $0 \leq \epsilon \leq \epsilon_0$, *there is a unique smooth solution of the 2-D water wave equation for a time period* $[0, e^{c/\epsilon}]$. *During this time, the solution remains smooth and small.*

Remark:

- Smallness in steepness $\partial_x y_0(x)$ of the initial interface and initial surface velocity are part of the assumption.

Theorem (3-D water wave)

Let $\xi_0 = (x, y, z_0(x, y))$ *be the initial interface,* $\mathfrak{u}_0 = \mathfrak{u}_0(x, y)$ *be the initial velocity along the interface* $\Sigma(0)$*. Assume that*

- $(\partial_x z_0, \partial_y z_0) = \epsilon\,(f(x, y), g(x, y))$ *(steepness)*
- $\mathfrak{u}_0 = \epsilon\, u(x, y)$

$(f, g) \in L^2(\Sigma(0))$*,* $u \in L^2(\Sigma(0))$*, and finitely many derivatives of* (f, g) *and* u *are in* L^2*.*

There is $\epsilon_0 > 0$*, such that if* $0 \leq \epsilon \leq \epsilon_0$*, there is a unique smooth solution of the 3-D water wave equation globally in time. During this time, the solution remains smooth and small.*

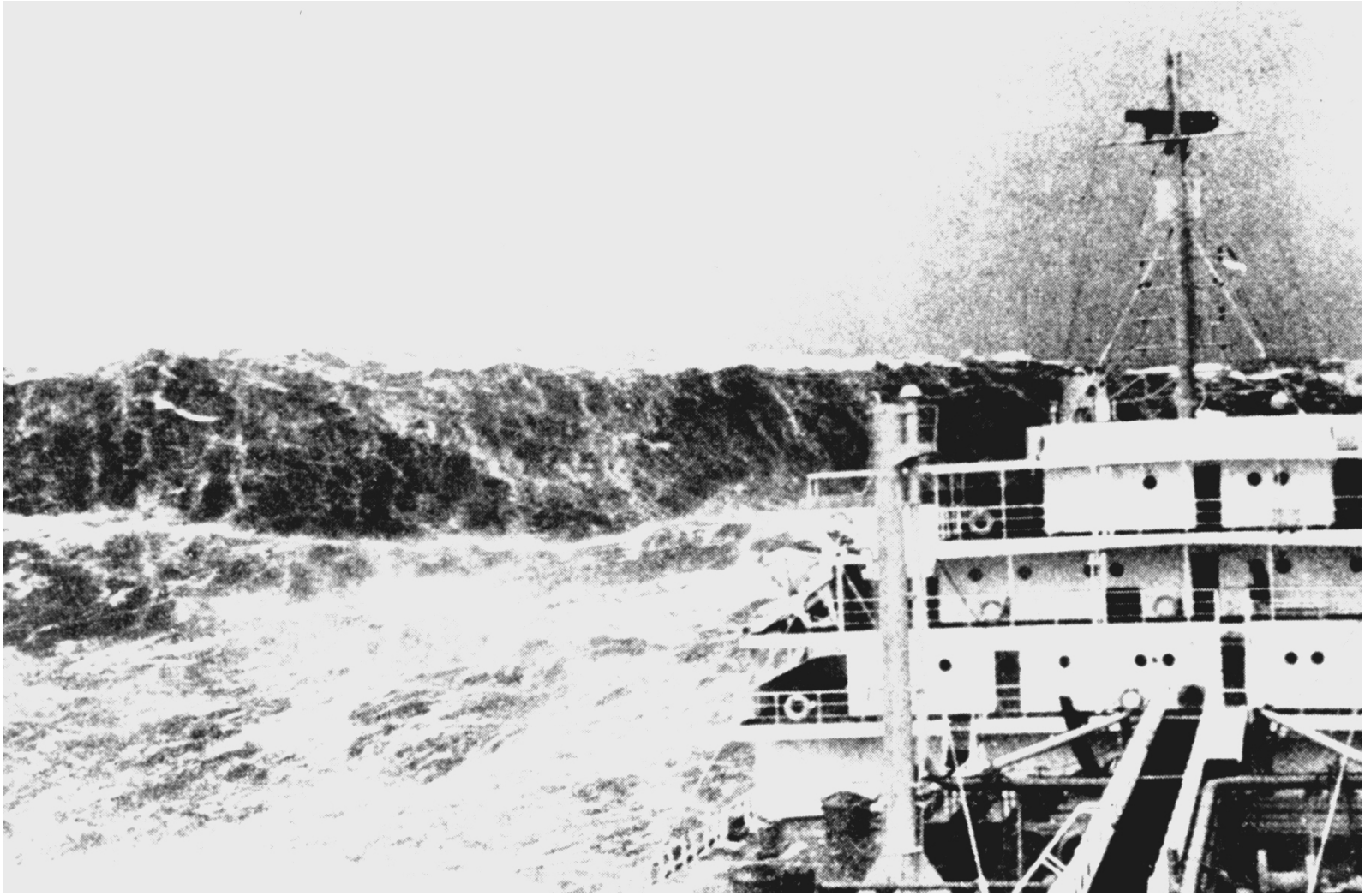
Remark:

- No smallness assumptions on the initial height function z, and the initial energy $\int_{\Omega(0)} |\mathbf{v}_0|^2$ can be infinite.

Remark:

- The total kinetic energy can be infinity in both the 2d and 3d cases: $\epsilon \times \infty$:
  2d: $\epsilon \times \infty$ in the constant wave motion direction;
  3d: $\epsilon \times \infty$ in depth.
- : The wave motion is localized in the L^2 sense.
- For 2-D water wave, if we only assume smallness in initial steepness and initial surface velocity and finitely many of their derivatives (as in the 3-D case), then the same proof as in the 3-D case gives us an existence for a time period $0 \leq t \leq 1/\epsilon^4$ for the 2-D water wave.

Open: sharpness of the 2-D result.



Rogue wave

Rogue waves is vastly massive waves (30m).

- Often appear in perfectly clear weather, without warning.
- It's exact causes are still unknown.

Possible causes

- Diffractive focusing: effect from coastline
- Focusing of currents
- Nonlinear effects

2-D water wave

key idea: reduce equation to a form from which we can derive results

Local existence:
Recall: the motion of the fluid is described by

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{g} - \nabla P, \qquad \Omega(t) \tag{4}$$

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 0, \qquad \Omega(t) \tag{5}$$

$$P = 0, \qquad \Sigma(t) \tag{6}$$

This is a nonlinear equation defined on moving domains.

Let

$$\Sigma(t) : z = x(\alpha, t) + i\, y(\alpha, t)$$

α: Lagrangian coordinate.

- $z_t =$ velocity $= \mathbf{v}(z(\alpha, t), t)$
- $z_{tt} =$ acceleration $= \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}$
- (4) $\iff$ $z_{tt} + i = -\nabla P$
- (6) $\Longrightarrow -\nabla P = i\mathfrak{a} z_\alpha$, ($\mathfrak{a} = -\dfrac{\partial P}{\partial \mathbf{n}} > 0$)
- (4), (6) $\iff z_{tt} + i = i\mathfrak{a} z_\alpha$
- (5) $\iff \overline{z}_t = \mathfrak{H}\overline{z}_t$. ($\overline{z}_t = \frac{1}{2}(I + \mathfrak{H})\overline{z}_t$.)
- $$\mathfrak{H} f(\alpha, t) = \frac{1}{\pi i} p.v. \int \frac{f(\beta, t) z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)}\, d\beta$$

Quasi-linear equation

$$z_{tt} + i = i\mathfrak{a} z_\alpha \tag{7}$$

$$\overline{z}_t = \mathfrak{H}\overline{z}_t \tag{8}$$

Take derivative to t to (7): we get

$$z_{ttt} - i\mathfrak{a} z_{t\alpha} = i\mathfrak{a}_t z_\alpha \tag{9}$$

(9) is a quasi-linear equation with the left hand side consisting of the higher order terms:

$$\mathfrak{a}_t |z_\alpha| \quad \sim \quad z_{tt},\ z_t \qquad \text{in regularity}$$

Let $\mathfrak{u} = z_t$,

$$\mathfrak{u}_{tt} + \mathfrak{a}\nabla_{\mathbf{n}}\mathfrak{u} = \frac{\mathfrak{u}_t + i}{|\mathfrak{u}_t + i|}\mathfrak{a}_t|z_\alpha| \tag{10}$$

$$(I + \mathfrak{K}^*)(\mathfrak{a}_t|z_\alpha|) = \Re\Big(\frac{iz_\alpha}{|z_\alpha|}\{2[\mathfrak{u}_t, \mathfrak{H}]\frac{\bar{\mathfrak{u}}_\alpha}{z_\alpha} + 2[\mathfrak{u}, \mathfrak{H}]\frac{\bar{\mathfrak{u}}_{t\alpha}}{z_\alpha}$$
$$- \frac{1}{\pi i}\int\Big(\frac{\mathfrak{u}(\alpha,t) - \mathfrak{u}(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2\bar{\mathfrak{u}}_\beta\, d\beta\}\Big)$$

where $\mathfrak{K}^*$: the adjoint of the double layered potential operator.

- We prove $\mathfrak{a} = -\dfrac{\partial P}{\partial \mathbf{n}} > 0$
- $\nabla_{\mathbf{n}}$ is a positive operator

Conclusion: (10) is of hyperbolic type $\Longrightarrow$ local well-posedness.

Global behavior in 2-D

Main ideas:

- Use a different coordinate system
- Use the dispersive aspect of the quasilinear equation

We find the quasilinear equation (10) is coordinate invariant:

- Notation:

$$U_k f(\alpha, t) = f \circ k(\alpha, t) = f(k(\alpha, t), t)$$

For fixed t, let $k = k(\alpha, t) : R \to R$ be a diffeomorphism, $k_\alpha > 0$,

$$k \circ k^{-1}(\alpha, t) = \alpha$$

- Define:

$$\zeta = z \circ k^{-1}, \quad u = z_t \circ k^{-1}, \quad w = z_{tt} \circ k^{-1}$$
$$b = k_t \circ k^{-1}, \qquad A \circ k = \mathfrak{a} k_\alpha$$

- Apply $U_{k^{-1}}$ to both sides of (10), we get

$$(\partial_t + b\partial_\alpha)^2 u - iA\partial_\alpha u = (\mathfrak{a}_t |z_\alpha|) \circ k^{-1} \frac{w+i}{|w+i|} \tag{11}$$

- where

$$(I + \mathcal{K}^*)((\mathfrak{a}_t|z_\alpha|) \circ k^{-1}) = \Re(\frac{i\zeta_\alpha}{|\zeta_\alpha|}\{2[w,\mathcal{H}]\frac{\bar{u}_\alpha}{\zeta_\alpha} + 2[u,\mathcal{H}]\frac{\bar{w}_\alpha}{\zeta_\alpha}$$
$$- \frac{1}{\pi i}\int(\frac{u(\alpha,t)-u(\beta,t)}{\zeta(\alpha,t)-\zeta(\beta,t)})^2 \bar{u}_\beta \, d\beta\})$$

Dispersive aspect:

Linearize (10) around the zero solution $\mathfrak{u} = 0$, we get free equation

$$\mathfrak{u}_{tt} + |D_\alpha|\mathfrak{u} = 0$$

- Decay rate:

$$|\mathfrak{u}(\cdot, t)|_{L^\infty} \leq \frac{c}{(t+1)^{1/2}} |\mathfrak{u}(\cdot, 0)|_{W^{1,1}}$$

(10) $\Longleftrightarrow$

$$\mathfrak{u}_{tt} + |D_\alpha|\mathfrak{u} = F(z, \mathfrak{u}, \mathfrak{u}_t, \mathfrak{u}_{tt}, \mathfrak{u}_\alpha)$$

- F consisting of the nonlinear terms. In particular, F contains quadratic terms.

Idea: look for possible cancellations in the quadratic terms.

The method of normal form: Let

$$\mathfrak{v} = \mathfrak{u} + K(\mathfrak{u})$$

Find an appropriate nonlinear form $K(\mathfrak{u})$ so that (hopefully) $\mathfrak{v}$ satisfies an equation

$$\mathfrak{v}_{tt} + |D_\alpha|\mathfrak{v} = F_1(z, \mathfrak{u}, \mathfrak{u}_t, \mathfrak{u}_{tt}, \mathfrak{u}_\alpha)$$

with F_1 consisting of only cubic and higher order terms, and

$$\|\mathfrak{v}\| \approx \|\mathfrak{u}\|$$

- Poincare
- Shatah: Klein-Gordon equation. Bilinear $K(\mathfrak{u})$ works. decay rate $1/t^{3/2}$.

Water wave

$$\mathfrak{v} = \mathfrak{u} + K(\mathfrak{u})$$

with a bilinear $K(\mathfrak{u})$ doesn't quite work for the water wave equation.

- There is a small divisor (resonance). However (from the right formulation of the water wave equation) we notice that it could be related to the coordinate system we use.

  Consider only the part without small divisor — the partial transformation:

- The partial transformation doesn't quite work, since it gives rise to an equation with bad structures (depending on $\mathfrak{u}$).
- The partial transformation and the resulting equation are not coordinate invariant.

- Let $z = x + iy = z(\alpha, t)$: equation of the interface at time t.
- We find the projection $(I - \mathfrak{H})$ (partially) works:
- The quantity

$$\Pi = (I - \mathfrak{H})(z - \bar{z}) = 2i(I - \mathfrak{H})y$$

satisfies the equation:

$$\begin{aligned}
&(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi \\
&= \frac{4}{\pi}\int \frac{(z_t(\alpha,t) - z_t(\beta,t))(y(\alpha,t) - y(\beta,t))}{|z(\alpha,t) - z(\beta,t)|^2} z_{t\beta}\, d\beta \\
&+ \frac{1}{\pi i}\int \left(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\right)^2 (z_\beta - \bar{z}_\beta)\, d\beta
\end{aligned} \tag{12}$$

- The right hand side of (13) consists of cubic & higher order terms.
- The structure of (13) is coordinate invariant.
- The left hand side of (13) still contains quadratic terms.
- $(I - \mathfrak{H})$ is fully nonlinear
- $\mathfrak{H} f(\alpha, t) = \int \frac{z_\beta(\beta,t)}{z(\alpha,t)-z(\beta,t)} f(\beta, t)\, d\beta$

Idea:

Look for an appropriate change of coordinates k.

- Chain rule:

$$U_{k^{-1}}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi = [(\partial_t + b\partial_\alpha)^2 - iA\partial_\alpha]\Pi \circ k^{-1}$$

- Q: Is there a change of variables k so that

$$b = k_t \circ k^{-1}, \quad A - 1 = \mathfrak{a}k_\alpha \circ k^{-1} - 1$$

  are quadratic?

Answer:

Yes.

- Let $h = h(\alpha, t) = \Psi(z(\alpha, t), t)$
- $\Psi(\cdot, t) : \Omega(t) \to P_-$ be the Riemann mapping.
- Let
$$k(\alpha, t) = 2x(\alpha, t) - h(\alpha, t)$$
then
$$b = k_t \circ k^{-1}, \quad A - 1 = \mathfrak{a} k_\alpha \circ k^{-1} - 1$$
are consisting of only quadratic and higher order terms.

Conclusion:

- The nonlinearity of the 2-D water wave equation is of cubic and higher orders when working in the right coordinate system with the right quantities.

Remark:
1. The projection $(I - \mathfrak{H})$ handles partly the bounded part in the formal bilinear normal form calculation. $(I - \mathfrak{H})$ is fully nonlinear and is coordinate invariant.
2. The coordinate change handles partly the small divisor (resonant) part in the formal bilinear normal form calculation.
3. Difference between $\partial_t^2 + |\partial_\alpha|$ and $\partial_t^2 - i\partial_\alpha$.

- Let ϕ be the velocity potential: $\nabla\phi = \mathbf{v}$, $(x, \eta(x,t))$ be the interface.
- The water wave equation is Hamiltonian in term of the variables $(\eta, \xi = \phi(x,\eta))$ (Zakharov).
- W. Craig et. al: Birkhoff normal form transformation procedure for periodic water wave (has discrete spectrum): there is no quadratic resonance in non-zero frequencies. (moment conditions can be imposed in the periodic case, so there is no need to consider the zero frequency.)

The resonance occurs at the zero frequency for our variables $(\eta, \mathbf{v}) = (\eta, \nabla\phi)$ (non-periodic):

- need to have a bound on $\phi = \nabla^{-1}\mathbf{v}$.
- In physically relevant situations there is no bound on ϕ.
- We resolve this difficulty by choosing the right coordinates.

Assume u, $k(x) - x$ are small:

$$u(k(x)) = u(x) + u'(x)(k(x) - x) + \dots \quad \text{Taylor expansion}$$

The norm of $u'(x)(k(x) - x)$ depends on the norms of u' and $k - x$. However

$$\int |u(k(x))|^2 \, dx = \int |u(x)|^2 \{k'(k^{-1}(x))\}^{-1} \, dx$$

$$\partial_x(u(k(x))) = u'(k(x))k'(x)$$

So the norm of $u(k(x))$ depends only on the norms of u and $k' - 1$.

If the norm of $k - x$ is not given, but only the norm of $k' - 1$ is given ....

If the transformation $u + K(u)$ with a bilinear $K(u)$ doesn't work, don't give up.

Germain, Masmoudi, Shatah (preprint July 2009) studied the 3D water wave global behavior assuming

- η & $|D|^{1/2}\phi \in W^{6,1}(R^2) \cap H^N(R^2) \cap L^2(x^2\, dx)$ with small norms.
  Such assumptions imply
- $\phi \in L^2(dx)$. In fact, $|\phi(x)| \lesssim 1/|x|^{3/2}$, as $x \to \infty$.
- The bilinear normal form for the variables (η, ϕ) has no small divisor (no resonance).
- However we know ϕ in general does not decay at spatial infinity on the interface, since "ϕ decays at infinity" is equivalent to " the line integral of the velocity field along any curve from the infinity to infinity is zero."
- This requires several moment conditions.
- the data set of GMS belongs to a closed lower dimensional subset of the data set we consider.

3-D water wave

- Use the framework of the algebra of quaternions.

  Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$, s. t.
- $e_i^2 = -1, e_i e_j = -e_j e_i, i \neq j, e_3 = e_1 e_2$.
- $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$
- $F : \Omega \subset \mathbb{R}^3 \to \mathcal{C}(V_2)$ is called Clifford analytic if $\mathcal{D}F = 0$.
- $F = \sum_{i=1}^{3} f_i e_i$ is Clifford analytic in Ω iff $\mathrm{div} F = 0$, $\mathrm{curl} F = 0$.
- F analytic in Ω iff $F = \mathfrak{H}F$, where
- $\mathfrak{H}g(\alpha, \beta) =$
  $p.v. \iint K(\xi(\alpha', \beta') - \xi(\alpha, \beta)) \, (\xi'_{\alpha'} \times \xi'_{\beta'}) \, g(\alpha', \beta') \, d\alpha' d\beta'$.
  $\partial\Omega : \xi = \xi(\alpha, \beta), \quad K(\xi) = -2\mathcal{D}\Gamma(\xi) = -\frac{2}{\omega_3} \frac{\xi}{|\xi|^3}$.

3-D water wave

The projection $(I - \mathfrak{H})$ still works.

- Let $\Sigma(t) : \xi = (x(\alpha, \beta, t), y(\alpha, \beta, t), z(\alpha, \beta, t))$ be the interface in Lagrangian coordinates (α, β),
- Let $\pi = (I - \mathfrak{H})z$.

$$
\begin{aligned}
&(\partial_t^2 - \mathfrak{a} N \times \nabla)\pi \\
&= \iint K(\xi' - \xi)\,(\xi_t - \xi_t') \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'})\overline{\xi'_t}\, d\alpha' d\beta' \\
&- \iint K(\xi' - \xi)\,(\xi_t - \xi_t') \times (\xi'_{t\beta'}\partial_{\alpha'} - \xi'_{t\alpha'}\partial_{\beta'}) z'\, d\alpha' d\beta' e_3 \\
&- \iint \partial_t K(\xi' - \xi)\,(\xi_t - \xi_t') \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) z'\, d\alpha' d\beta' e_3
\end{aligned}
\tag{13}
$$

Coordinate change

$k = \xi - (I + \mathfrak{H})ze_3 + \mathfrak{K}ze_3$ Here $\mathfrak{K}$ is the double layered potential operator.

Applying coordinate change k^{-1} gives for $\chi = \pi \circ k^{-1}$,

$$
\begin{aligned}
&((\partial_t + b \cdot \nabla_\perp)^2 - A\mathcal{N} \times \nabla)\chi \\
&= \iint K(\zeta' - \zeta)\,(u - u') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\overline{u'}\, d\alpha' d\beta' \\
&- \iint K(\zeta' - \zeta)\,(u - u') \times (u'_{\beta'}\partial_{\alpha'} - u'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\, d\alpha' d\beta' e_3 \\
&- \iint ((u' - u) \cdot \nabla)K\,(u - u') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\, d\alpha' d\beta' e_3
\end{aligned}
$$

- The first term on the right hand side is of type $(I + \mathfrak{H})Q$,
- Q quadratic. The left hand side is of type $(I - \mathfrak{H})$, or is analytic in the air region.
- The projection $(I - \mathfrak{H})$ applies to the first term on the right turns it into cubic.
  Let $\zeta = \xi \circ k^{-1} = P + \lambda$, $P = (\alpha, \beta)$.
- The left hand side = $\partial_t^2 \chi - \mathcal{N} \times \nabla \chi +$cubic terms = $\partial_t^2 \chi - (e_2 \partial_\alpha - e_1 \partial_\beta)\chi - \partial_\beta \lambda \partial_\alpha \chi + \partial_\alpha \lambda \partial_\beta \chi +$cubic terms.

- $\mathcal{N} \times \nabla\chi = (e_2\partial_\alpha - e_1\partial_\beta)\chi + \partial_\beta\lambda\partial_\alpha\chi - \partial_\alpha\lambda\partial_\beta\chi$.
- $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ is a null form.

- $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ is one of the null forms treated by Klainerman for the wave equation.
- Can $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ be transformed away by a transformation that is meaningful in physical sense? since although $(I - \mathfrak{H})$ as a projection is clear and concise in mathematical sense, it's physical meaning remained unexplained.
- Answer: $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ cannot be transformed away by a transformation concise in the physical space.

- Use method of vector fields.
- For 2D water wave equation: the operator $\partial_t^2 - i\partial_\alpha$ has invariant vector fields ∂_t, ∂_α, $L_0 = \frac{1}{2}t\partial_t + \alpha\partial_\alpha$, $\Omega_0 = \alpha\partial_t + \frac{1}{2}ti$.
- For 3D water wave equation: the operator $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$ has invariant vector fields ∂_t, $\partial_\alpha, \partial_\beta$, $L_0 = \frac{1}{2}t\partial_t + \alpha\partial_\alpha + \beta\partial_\beta$, $\varpi = \alpha\partial_\beta - \beta\partial_\alpha - \frac{1}{2}e_3$ and Ω_{01}, Ω_{02}.
- Combine a $L^2 \to L^\infty$ decay estimate and the energy estimates we prove the energy remains bounded for all time for 3D water wave and almost all time for 2D water wave.

We get almost global wellposedness for 2-D water wave (July 2009, Inventiones),
and global wellposedness for 3-D water wave.

The complete statement of global well-posedness for 3D water wave equation.

Let $s \geq 27$, $\max\{[\frac{s}{2}]+1, 17\} \leq l \leq s-10$.

- Assume that initially the interface is a graph

$$\xi(\alpha,\beta,0) = \xi^0 = (\alpha,\beta,z^0(\alpha,\beta)),$$

- initial velocity $\xi_t(\cdot,0) = \mathfrak{u}^0(\cdot)$, acceleration $\xi_{tt}(\cdot,0) = \mathfrak{w}^0(\cdot)$,
- $(\xi^0, \mathfrak{u}^0, \mathfrak{w}^0)$ satisfy the compatibility condition.

Let $\Gamma = \partial_\alpha,\ \partial_\beta,\ \alpha\partial_\alpha + \beta\partial_\beta,\ \alpha\partial_\beta - \beta\partial_\alpha$. Assume that

- $L = \sum_{\substack{|j|\leq s-1 \\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j|D|^{1/2}z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j\partial z^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{u}^0\|_{H^{3/2}(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{w}^0\|_{H^1(\mathbb{R}^2)} < \infty$
- $\epsilon = \sum_{\substack{|j|\leq l+3 \\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j|D|^{1/2}z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j\partial z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{u}^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{w}^0\|_{L^2(\mathbb{R}^2)}$

Theorem (3D water wave problem)

There exists $\epsilon_0 > 0$*, such that for* $0 \leq \epsilon \leq \epsilon_0$*, the initial value problem of the 3D water wave equation has a unique classical solution globally in time. For each time* $0 \leq t < \infty$*, the interface is a graph, the solution has the same regularity as the initial data and remains small, and the* L^∞ *norm of the solution decay at the rate* $\frac{1}{t}$*.*

Conclusion:

- The effect of the projection $(I - \mathfrak{H})$ is to clean up the equation so that the terms should not be there doesn't appear in this proper setting.
- the coordinate change $k = \xi - (I + \mathfrak{H})ze_3 + \mathfrak{K}ze_3$ is compatible with this projection.

Further questions:

- Characterize the largest class of data where no blow-up occurs.
- Identify data that gives rise to blow-up at finite time.