# Math 122L

## Additional Homework Problems

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# Contents

# Review of AP AB Differentiation Topics

1. Let $f(x) = x^2 + 4$.
   (a) Find the average rate of change over the interval [1,2].
   (b) Find the average rate of change over the interval [1,1.5].
   (c) Find the average rate of change over the interval [1,1.1].
   (d) Find the instantaneous rate of change at $x = 1$.

2. Suppose $f$ is an invertible function such that both $f$ and $f^{-1}$ are differentiable. Recall that $f\left(f^{-1}(x)\right) = x$. Use implicit differentiation to find a formula for $\frac{d}{dx}\left(f^{-1}(x)\right)$.

3. Suppose $f(1) = 2$, $f'(1) = 3$, $f^{-1}(1) = 1$, $g(1) = 1$, $g'(1) = 4$, and $g''(1) = 5$. Find the derivative of the following functions at $x = 1$:
   (a) $\sqrt{f(x)}$
   (b) $f\left(\sqrt{x}\right)$
   (c) $(g(x))^2$
   (d) $2^{g(x)}$
   (e) $e^{f(x)g(x)}$
   (f) $e^{f(g(x))}$
   (g) $\dfrac{g(x)}{g'(x)}$
   (h) $f^{-1}(x)$.

4. Let $f(x) = ax^2 + bx + c$. Suppose that $f(1) = 7$, and that the slope of the tangent lines to $f$ at $x = 2$ and $x = 4$ are 12 and 20, respectively. Find $a$, $b$, and $c$.

5. Use the line tangent to $f(x) = \sqrt[3]{1 + 3x}$ at $x = 0$ to estimate $\sqrt[3]{1.03}$..

6. If $f(x) = \lim_{t \to x} \dfrac{\sec(t) - \sec(x)}{t - x}$, find $f'\left(\dfrac{\pi}{4}\right)$.

7. For the following, assume $a$, $b$, and $c$ are positive constants.
   (a) Express $\ln(a + b) + \ln(a - b) - 2\ln(c)$ as a single logarithm.
   (b) Simplify $\left(\dfrac{3a^{1/2}b}{a^2 b^{-1/2}}\right)^{-2}$ so that there are no negative exponents.

8. Suppose $f$ is a twice differentiable function and that $f'(x)$ has one root.
   (a) How many roots can $f(x)$ have?
   (b) How many roots can $f''(x)$ have?

9. (a) Suppose $y = f(x)$ is a **linear** function such that increasing $x$ by 1 increases $y$ by 5. Then increasing $x$ by 2 increases $y$ by _____.

(b) Suppose $y = f(x)$ is an **exponential** function such that increasing $x$ by 1 increases $y$ by a factor of 5. Then increasing $x$ by 2 increases $y$ by a factor of _____.

10. Find and correct the mistakes in the following:

(a) $x^2 + 3x + 2 = 3 \Longrightarrow (x+2)(x+1) = 3 \Longrightarrow x = -2, x = -1$

(b) $x^2 + 1 = x + 1 \Longrightarrow x^2 = x \Longrightarrow x = 1$

(c) $(x+2)^2 = 4 \Longrightarrow x^2 + 2^2 = 4 \Longrightarrow x^2 = 0 \Longrightarrow x = 0$

(d) $2^a 2^b = 32 \Longrightarrow 2^{ab} = 2^5 \Longrightarrow ab = 5$

(e) $(2^a)^2 = 16 \Longrightarrow 2^{a^2} = 2^4 \Longrightarrow a = \pm 2$

(f) $\sin^{-1}(x) = 2 \Longrightarrow \dfrac{1}{\sin(x)} = 2 \Longrightarrow \sin(x) = \dfrac{1}{2} \Longrightarrow x = \dfrac{\pi}{6}$

(g) $\dfrac{1}{1+x} = 2 \Longrightarrow 1 + \dfrac{1}{x} = 2 \Longrightarrow \dfrac{1}{x} = 1 \Longrightarrow x = 1$

# L'Hopital's Rule and Relative Rates of Growth

1. Suppose $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, $\lim_{x\to a} r(x) = \infty$, and $\lim_{x\to a} s(x) = \infty$. For each of the following limits, decide whether or not it would be appropriate to use L'Hopital's Rule.

(a) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$

(b) $\lim_{x\to a} \dfrac{f(x)}{s(x)}$

(c) $\lim_{x\to a} f(x) - g(x)$

(d) $\lim_{x\to a} (f(x))^{g(x)}$

(e) $\lim_{x\to a} (r(x))^{f(x)}$

(f) $\lim_{x\to a} f(x)s(x)$

2. Find the mistake(s) in each of the following. Then solve the given limit correctly:

(a) $\lim_{x\to 0} \dfrac{\sin(x)}{x} = \lim_{x\to 0} \dfrac{x\cos(x) - \sin(x)}{x^2} = 0$

(b) $\lim_{x\to 0} \dfrac{\cos(x)}{x} = \lim_{x\to 0} \dfrac{-\sin(x)}{1} = 0$

(c) $\lim_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x = (1)^\infty = 1$

3. Suppose $f(4) = 0$ and $f'(4) = 2$. Evaluate the following:

(a) $\lim_{x\to 0} \dfrac{f(4+2x) + f(4+x)}{x}$

(b) $\lim_{x\to 0} \frac{f(4+x)-f(4-x)}{2x}$

4. For which values of $a$ and $b$ is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin(4x)}{x^3}+a+\frac{b}{x^2}\right)=0$$

# Riemann Sums

1. Draw a function, $f(x)$, in which the LHS(2) approximation of $f(x)$ on $[0,2]$ is more accurate than the MPS(2) approximation.

2. For which class of functions are the left-hand and right-hand sums exact? Trapezoid rule?

3. Consider the region between $y=1$, $y=e^{-x^2}$, and the $x=2$. Estimate the area of this region using a right-hand sum with 4 rectangles.

# Definition of the Definite Integral

1. If $\sum_{k=r}^{s} f\left(-3+\frac{k}{2}\right)\left(\frac{1}{2}\right)$ is the left-hand Riemann sum, with $n=8$ rectangles, that approximates $\int_{-2}^{2} f(x)\,dx$, find $r$ and $s$.

2. Solve $\int_1^2 \left(x^2+x+1\right)\,dx$ using the definition of the definite integral. Note that $\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6}$.

3. Suppose function $f$ passes through the following points:

| $x$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2 | 1 | -1 | 2 | 5 | 8 | 5 |

(a) Approximate $\int_0^{12} xf(x)\,dx$ using a Left-Hand Riemann sum with 6 rectangles.

(b) Approximate $\int_0^{12} xf(x)\,dx$ using a Right-Hand Riemann sum with 3 rectangles.

(c) Approximate $\int_0^{12} xf(x)\,dx$ using a Midpoint Riemann sum with 3 rectangles.

4. Consider a continuous function $f(x)$. Using a Right-Hand Riemann sum, we could approximate $\int_1^{10} f(x)\,dx$ by $\sum_{k=1}^{10} f\left(1+\frac{9k}{10}\right)\left(\frac{9}{10}\right)$. If we instead want to approximate $\int_{11}^{20} f(x)\,dx$ with the same number of rectangles, how should we adjust the Riemann sum?

# MVT and FTC Part I

1. Evaluate the following limits:

   (a) $\lim_{n\to\infty} \sum_{k=0}^{n-1} \sec^2\left(\frac{-\pi}{4} + \frac{k\pi}{2n}\right)\frac{\pi}{2n}$

   (b) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{\sqrt{1-\frac{k^2}{4n^2}}}\right)\frac{1}{2n}$

2. Without using a calculator (or Maple), rank the following quantities from smallest to largest:

   $\int_0^1 e^x\,dx, \qquad \sum_{k=1}^{10} \exp\left(\frac{(k-1)+(k)}{20}\right)\frac{1}{10}, \qquad \sum_{k=1}^{100} \exp\left(\frac{(k-1)+(k)}{200}\right)\frac{1}{100}$

3. Evaluate $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$.

4. The following statements are FALSE. Prove this by providing a counterexample in each case.

   (a) For any function $f(x)$, $\int_0^1 |f(x)|\,dx = \left|\int_0^1 f(x)\,dx\right|$.

   (b) For any functions $f(x)$ and $g(x)$, $\int_0^1 f(x)g(x)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx$.

   (c) For any positive function $f(x)$, $\int_0^1 \sqrt{f(x)}\,dx = \sqrt{\int_0^1 f(x)\,dx}$.

5. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2k^2}{n^3}$ by:

   (a) using Riemann sums with $\Delta x = \frac{1}{n}$

   (b) using Riemann sums with $\Delta x = \frac{2}{n}$

6. Use Riemann sums to prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \ln(2)$.

# FTC Part II

1. Suppose $f(x) = \int_0^x \left(\int_1^{\sin(t)} \sqrt{1+u^4}\,du\right)\,dt$.

   (a) Is $f$ increasing or decreasing at $x = \pi$?

   (b) Find $f''(x)$.

2. Find a function $f$ such that $x^2 = 1 + \int_1^x \sqrt{1 + (f(t))^2}\, dt$ for all $x > 1$.

3. Find a function $f(x)$, such that $f'(x) = \sin\left(e^{x^2}\right)$ and $f(2) = 4$.

## U-Substitution

1. Let $f(x)$ be a continuous function. Evaluate $\int_{\pi/2}^{3\pi/2} f(\cos(x))\sin(x)\, dx$.

2. Let $f(x) = \dfrac{\ln(x)}{x}$.

   (a) Find the average value of $f(x)$ on $\left[\dfrac{1}{2}, 2\right]$.

   (b) Find a value $\dfrac{1}{2} \leq c \leq 2$ at which $f(x)$ equals its average value.

3. Suppose $f(x)$ is an even function such that $\int_{-1}^{1} f(x)\, dx = 4$. Find $\int_{-2}^{-1} 3f(x+2)\, dx$.

## Integration by Parts

1. Evaluate $\int \cos^2(x)\, dx$ by:

   (a) using the trig identity $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$

   (b) using integration by parts

2. Evaluate the following:

   (a) $\int \cos(x)e^x\, dx$

   (b) $\int \cos(x)e^{\sin(x)}\, dx$

3. Suppose $f(0) = 0 = g(0)$, $f(2) = 1$, $f'(2) = 2$, $g(2) = 3$, $g'(2) = 4$, and $\int_0^2 f''(x)g(x)\, dx = 5$. Find $\int_0^2 f(x)g''(x)\, dx$.

# Improper Integrals

1. Evaluate $\displaystyle\int_0^\infty x^2 e^{-x^2}\,dx$, given that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

2. Determine whether each of the following integrals converge or diverge by using the Comparison Theorem using the suggested comparison:

   (a) $\displaystyle\int_1^\infty \frac{1}{x^3+1}\,dx$, comparing with $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

   (b) $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$, comparing with $\displaystyle\int_1^\infty \frac{2}{x}\,dx$

   (c) $\displaystyle\int_0^\pi \frac{\sin^2(x)}{\sqrt{x}}\,dx$, comparing with $\displaystyle\int_0^\pi \frac{1}{\sqrt{x}}\,dx$

3. Show that $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ converges. Why can we not use this integral with the Comparison Theorem in order to show that $\displaystyle\int_0^\infty \frac{\arctan(x)}{2+e^x}\,dx$ converges? How can we adjust $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ so that it is useful with the Comparison Theorem for $\displaystyle\int_0^\infty \frac{\arctan(x)}{2+e^x}\,dx$?

4. Evaluate the following integrals, or show that they diverge. Make sure to show all associated work.

   (a) $\displaystyle\int_2^6 \frac{1}{\sqrt{x-2}}\,dx$

   (b) $\displaystyle\int_0^\pi \frac{\sin(x)}{\cos(x)}\,dx$

   (c) $\displaystyle\int_{-1}^1 \frac{e^x}{e^x-1}\,dx$

   (d) $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

5. Suppose that $f$ is a positive, continuous function such that $\displaystyle\int_0^\infty f(x)\,dx$ converges, and $a$ is a positive number. Decide whether the following must be true:

   (a) $\displaystyle\int_0^\infty af(x)\,dx$ converges

   (b) $\displaystyle\int_0^\infty f(ax)\,dx$ converges

   (c) $\displaystyle\int_0^\infty f(a+x)\,dx$ converges

   (d) $\displaystyle\int_0^\infty \big(a+f(x)\big)\,dx$ converges

# Partial Fractions

1. Evaluate the following:

   (a) $\displaystyle\int \frac{1}{1-x}\,dx$

   (b) $\displaystyle\int \frac{x}{1-x}\,dx$

   (c) $\displaystyle\int \frac{1}{1-x^2}\,dx$

   (d) $\displaystyle\int \frac{x}{1-x^2}\,dx$

   (e) $\displaystyle\int \frac{1}{1+x^2}\,dx$

   (f) $\displaystyle\int \frac{1}{1+9x^2}\,dx$

   (g) $\displaystyle\int \frac{1}{9+x^2}\,dx$

   (h) $\displaystyle\int \arctan(x)\,dx$

   (i) $\displaystyle\int \frac{x}{e^{-x}}\,dx$

   (j) $\displaystyle\int \frac{1}{1+e^{-x}}\,dx$

# Introduction to Probability

1. Two events $A$ and $B$ are said to be **mutually exclusive** if the probability that they both occur is zero.

   (a) Suppose you roll a fair six-sided die. Give an example of two events that are mutually exclusive.

   (b) Let $A$ and $B$ be events such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Prove that it is impossible for $A$ and $B$ to be both independent and mutually exclusive.

2. Show that $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for any two events $A$ and $B$.

3. Suppose you row two fair $n$-sided dice. Find the probability of each of the following events:

   (a) the maximum of the two numbers rolled is less than or equal to 4.

   (b) the maximum of the two numbers rolled is less than or equal to 5.

   (c) the maximum of the two numbers rolled is less than or equal to $k$, where $k \in \{1, 2 \ldots, n\}$.

   (d) the maximum of the two numbers rolled is exactly equal to $k$, where $k \in \{1, 2 \ldots, n\}$.

4. Suppose you roll 2 fair six-sided dice. A list of possible outcomes is provided below.

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}$$

Let $A$ be the event that the second roll is greater than the first roll. Let $B$ be the event that you roll two numbers whose sum is $\leq 4$.

(a) Find $\mathbb{P}(A)$ and $\mathbb{P}(B)$

(b) Are $A$ and $B$ independent? Justify your answer mathematically.

(c) Suppose you roll the pair of dice 4 times and count the number of times that either event $A$ *or* event $B$ occurs. What is the probability that this happens 25% of the time?

(d) Suppose you get to choose a number and if the sum of the two die rolls equals that number, you win a prize. What number should you choose?

# Expected Value

1. Suppose $X$ is a random variable with just two possible values $a$ and $b$. Find a formula for $\mathbb{P}(X = a)$ and for $\mathbb{P}(X = b)$ in terms of only $a$, $b$, and $\mu = \mathbb{E}[X]$.

2. Consider the experiment of flipping a biased coin (which comes up heads with probability $\frac{3}{5}$) four times. Find the expected value of each of the following random variables.

   (a) $X$ is the number of heads.

   (b) $Y$ is the number of heads minus the number of tails.

   (c) $Z$ is equal to $|Y|$

3. If $a$ and $b$ are constants and $X$ is a random variable, show that $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$.

4. A box contains 5 marbles; 2 are labeled with the number 1 and 3 are labeled with the number 2. Suppose you reach in and select two marbles, without replacement. Let $X$ be the product of the two numbers drawn. Find $\mathbb{E}[X]$.

# Introduction to Sequences and Series

1. Let $a_k = e^{-k} + 1$

   (a) Does $\{a_k\}_{k=1}^{\infty}$ converge or diverge? Explain.

   (b) Does $\displaystyle\sum_{k=1}^{\infty} a_k$ converge or diverge? Explain.

2. Fill in the blank: $\sum_{k=1}^{\infty} a_k = \sum_{k=10}^{\infty} a_{\_\_\_}$

3. Suppose $\sum_{n=1}^{\infty} a_n$ converges and that $a_n \neq 0$ for all $n \geq 1$. Show that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

4. Use partial fraction decomposition to show that $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ converges, and find its sum.

5. Determine whether $\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^{k^2}$ converges or diverges.

6. The series $\sum_{k=1}^{\infty} a_k$ has partial sums $S_n$ defined by

$$S_n = S_{n-1} + \cos(S_{n-1}) \qquad S_1 = 1$$

   Suppose this series converges to a finite number, $L$ where $0 < L < 4$.

   (a) Find $\lim_{k \to \infty} a_k$.

   (b) Find $\sum_{k=1}^{\infty} a_k$.

7. Consider the sequence $\{a_k\}_{k=1}^{\infty}$, where $a_k = \frac{1}{k}$.

   (a) Draw a plot of this sequence, together with the graph of the function $f(x) = \frac{1}{x}$. To draw the sequence, draw rectangles with width one, and height $a_k$.

   (b) Use your graph to determine which of the following relations is correct:
   $\sum_{k=1}^{n} \frac{1}{k} \leq \int_1^n \frac{1}{x}\,dx$ $\qquad$ $\sum_{k=1}^{n} \frac{1}{k} = \int_1^n \frac{1}{x}\,dx$ $\qquad$ $\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^n \frac{1}{x}\,dx$

   (c) Find $\lim_{n \to \infty} \int_1^n \frac{1}{x}\,dx$.

   (d) What can you concludes about the convergence/divergence of $\sum_{k=1}^{\infty} a_k$?

# Probability and Geometric Series

1. Find the sum of $\sum_{k=1}^{\infty} \frac{1}{e^{2k-1}}$.

2. Find two divergent series, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $\sum_{k=1}^{\infty} (a_k + b_k)$ converges.

3. Find two convergent series, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $\sum_{k=1}^{\infty} \left(\frac{a_k}{b_k}\right)$ diverges.

4. Evaluate the following limits.

   (a) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(e^{1+\frac{2k}{n}}\right)\left(\frac{2}{n}\right)$

   (b) $\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{e}\right)^{k+1}\left(\frac{e}{2}\right)^{k}$

5. What restrictions, if any, must be placed on $a$, $b$, and $c$, for the series $\sum_{k=4}^{\infty} \left(a^{2k-3} b c^{-k/2}\right)$ to converge? In the case that it does converge, find its sum.

## Integral Test

1. Consider the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$.

   (a) Draw the graph of $f(x) = \frac{1}{2^x}$ for $0 \leq x \leq 10$.

   (b) On your graph from (a), draw rectangles that represent $\sum_{k=1}^{10} \frac{1}{2^k}$ and indicate that $\sum_{k=1}^{10} \frac{1}{2^k} \leq \int_0^{10} \frac{1}{2^x}\,dx$.

   (c) Use the Integral Test to show that $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

   (d) Find the sum of $\sum_{k=1}^{\infty} \frac{1}{2^k}$.

2. For each of the following series, determine why the Integral Test cannot be used.

   (a) $\sum_{k=1}^{\infty} \frac{1}{k!}$

   (b) $\sum_{k=1}^{\infty} \arctan(k)$

   (c) $\sum_{k=1}^{\infty} \sin(n)$

3. Suppose you approximate $\sum_{k=1}^{\infty} e^{-k}$ by its 10th partial sum.

(a) Use the Integral Test error bounds to find both an upper and a lower bound on the error in this approximation.

(b) Without using a calculator, find the exact value of the error in this approximation.

## Comparison Tests

1. Give an example of:

   (a) $a_k$ and $b_k$ such that $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$, $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges, and $\displaystyle\sum_{k=1}^{\infty} b_k$ converges.

   (b) $a_k$ and $b_k$ such that $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = 0$, $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, and $\displaystyle\sum_{k=1}^{\infty} b_k$ diverges.

2. The Comparison Test requires that the terms of both series in the comparison be nonnegative. To show why, give an example of $a_k > 0$ and $b_k < 0$ such that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges but $\displaystyle\sum_{k=1}^{\infty} b_k$ diverges (even though $b_k < a_k$).

3. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$.

   (a) Use the Comparison Test to show that this series converges.

   (b) Note that this series satisfies the conditions of the Integral Test. Thus, we can use the associated error bounds to say that if we approximate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$ by its 10th partial sum, the resulting error is bounded above by $\displaystyle\int_{10}^{\infty} \frac{1}{x^3+1}\,dx$. The value of this integral is difficult to find but we know that it is bounded above by $\displaystyle\int_{10}^{\infty} \frac{1}{x^3}\,dx$. Use this to find an upper bound on the error.

4. Determine whether the following series converge or diverge:

   (a) $\displaystyle\sum_{k=1}^{\infty} \ln(k)$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\ln(k)}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$

   (d) $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{1}{k}\right)$

(e) $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$

## Alternating Series and Absolute Convergence

1. Suppose that the series $\sum_{k=1}^{\infty} a_k$ converges and that $a_k > 0$ for all $k \geq 1$. Decide whether the following series converge or diverge, and explain why.

   (a) $\sum_{k=1}^{\infty} \frac{a_k}{k}$

   (b) $\sum_{k=1}^{\infty} \frac{1}{a_k}$

   (c) $\sum_{k=1}^{\infty} a_k^2$

   (d) $\sum_{k=1}^{\infty} (-1)^k a_k$

2. Consider the series $\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{4^2} - \frac{1}{4^3} + \frac{1}{5^2} - \frac{1}{5^3} + \cdots$. Why can we not use the Alternating Series Test here? Determine whether the series converges or diverges.

3. Consider the series $\frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} - \frac{1}{9^2} + \cdots$. Why can we not use the Alternating Series Test here? Determine whether the series converges or diverges.

4. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$. For which values of $p$ does this series:

   (a) converge absolutely?

   (b) converge conditionally?

   (c) diverge?

5. Find an upper bound on the error incurred when using:

   (a) $\sum_{k=1}^{10} \frac{1}{k^2}$ to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

   (b) $\sum_{k=1}^{10} \frac{(-1)^k}{k^2}$ to approximate $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$.

6. What is wrong with the following arguments?

(a) Because $\lim_{k\to\infty} \frac{k}{2k+1} \neq 0$ and $\frac{(k+1)}{2(k+1)+1} \nleq \frac{k}{2k+1}$, the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k+1}$ diverges by the Alternating Series Test.

(b) Because $\frac{\cos(k)}{k^2+1} \leq \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (as a p-series with $p = 2 > 1$), the series $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^2+1}$ converges by the Comparison Test.

# Ratio Test

1. If $a_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{a_{k+1}} = 2$, find $\lim_{k\to\infty} a_k$.

2. Let $0 < p, q < 1$. Why can't the Ratio Test be used on $p + q + p^2 + q^2 + p^3 + q^3 + \cdots$? Show that this series converges, and find its sum.

3. Consider the series $\sum_{k=1}^{\infty} \frac{x^k}{k}$.

   (a) Use the Ratio Test to show that this series converges for $|x| < 1$.

   (b) Note that the Ratio Test gives no information for $x = \pm 1$. Use other methods to determine whether or not the series converges at these two values of $x$.

# Probability Distributions and Expected Value

1. Consider the following function:

$$f(x) = \begin{cases} 0 & \text{if } -\infty < x < -1 \\ ax & \text{if } -1 \leq x < 0 \\ bx^3 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x < \infty \end{cases}$$

   (a) Find the values of constants $a$ and $b$ that make $f(x)$ a probability density function with $\mathbb{E}[X] = \frac{4}{15}$.

   (b) With the constants you found in (a), find the median of this distribution.

# Normal Distributions

1. Let $X$ be a random variable that is normally distributed with mean 10 and standard deviation 2. Solve the following without a calculator.

   (a) If $\mathbb{P}(X > a) = 0.1$, then decide whether the following are true or false:

- $a > 10$
- $a > 12$
- $a > 14$

(b) Find $\mathbb{P}(6 \leq X \leq 12)$.

# Power Series

1. Determine whether each of the following is a power series.

   (a) $\displaystyle\sum_{k=0}^{\infty} x^{-k}$

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

   (c) $\displaystyle\sum_{k=0}^{\infty} k^x$

   (d) $\displaystyle\sum_{k=0}^{\infty} (x-k)^2$

   (e) $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k}$

2. Suppose we know $\displaystyle\sum_{k=0}^{\infty} c_k x^k$ has radius of convergence 2.

   (a) What is $\displaystyle\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|}$?

   (b) What is the radius of convergence of $\displaystyle\sum_{k=0}^{\infty} c_k (x-1)^k$?

   (c) What is the radius of convergence of $\displaystyle\sum_{k=0}^{\infty} c_k x^{2k}$?

3. Find a power series that has interval of convergence:

   (a) $(1, 3)$

   (b) $[1, 3)$

   (c) $(1, 3]$

   (d) $[1, 3]$

# Representing Functions as Power Series

1. Find the mistake(s) in the following:

   (a) $\dfrac{1}{(1+x)^2} = \left(\dfrac{1}{1+x}\right)^2 = \left(\displaystyle\sum_{k=0}^{\infty}(-1)^k x^k\right)^2 = \displaystyle\sum_{k=0}^{\infty} x^{2k}$

   (b) $\dfrac{d}{dx}\left(\displaystyle\sum_{k=0}^{\infty}(3x)^k\right) = \displaystyle\sum_{k=0}^{\infty} k(3x)^{k-1}$

   (c) $\displaystyle\int \sum_{k=0}^{\infty}(-1)^k x^k \, dx = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1}$

# Taylor Polynomials

1. Find the Taylor polynomial, centered at $x = a$, of degree $n$ for each of the following functions (you can use these derivations for the homework from section 8.8):

   (a) $f(x) = \sin(x)$, $a = \pi/6$, $n = 4$

   (b) $f(x) = e^{x^2}$, $a = 0$, $n = 3$

   (c) $f(x) = \ln(1 + 2x)$, $a = 1$, $n = 3$

   (d) $f(x) = x\sin(x)$, $a = 0$, $n = 4$

   (e) $f(x) = x\ln(x)$, $a = 1$, $n = 3$

2. Give an example of a function $f(x)$, such that the Taylor polynomial of degree 4 of $f$ is the same as the Taylor polynomial of degree $n$ for all $n > 4$.

3. The table below gives information about a continuous function $f(x)$:

| $f(0)$ | $f'(0)$ | $f''(0)$ | $f'''(0)$ | $f^{(4)}(0)$ |
|---|---|---|---|---|
| 0 | 1 | -3 | 7 | -15 |

   (a) Use a 4th degree Taylor polynomial to estimate $f(0.1)$.

   (b) Use a 4th degree Taylor polynomial to estimate $\displaystyle\int_0^{0.5} f(x)\,dx$.

# Taylor Series

1. Find a power series representation for $\ln(1 + x)$ centered about $x = 0$ in two different ways:

   (a) by relating it back to the function $\dfrac{1}{1-x}$

   (b) by deriving its Taylor series

2. Use Taylor series to find the 10th derivative of $f(x) = \sin(x^2)$ at $x = 0$.

3. Find the sum of $\displaystyle\sum_{k=1}^{\infty} \frac{ke^{-2}2^{k-1}}{k!}$

4. Let $f(t) = te^t$.

   (a) Find the Taylor series for $f(t)$ centered at $t = 0$.

   (b) Use your answer to (a) to find the Taylor series representation, about $x = 0$, for $\displaystyle\int_0^x f(t)\,dt$.

   (c) Use part (b) to prove that $\displaystyle\frac{1}{2} + \frac{1}{3} + \frac{1}{4(2!)} + \frac{1}{5(3!)} + \frac{1}{6(4!)} + \cdots = 1$.

# Fourier Series Preparation

1. Use Maple to compute each of the following for various integers $m$ and $n$:

   (a) $\displaystyle\int_{-\pi}^{\pi} a\,dx$

   (b) $\displaystyle\int_{-\pi}^{\pi} \sin(mx)dx$

   (c) $\displaystyle\int_{-\pi}^{\pi} \cos(mx)dx$

   (d) $\displaystyle\int_{-\pi}^{\pi} \sin^2(mx)dx$

   (e) $\displaystyle\int_{-\pi}^{\pi} \cos^2(mx)dx$

   (f) $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\sin(mx)dx$

   (g) $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx$

   (h) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$

   (i) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx$

# Fourier Series

1. Give an example of a function, $f(x)$, such that the Fourier series for $f(x)$ is exactly equal to $f(x)$.

2. Suppose $f(x)$ has Fourier series

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin\left((2k-1)x\right)$$

(a) What is the period of $f$?

(b) What is the average value of $f(x)$ on the interval $[-\pi, \pi]$?

(c) What is $\int_{-\pi}^{\pi} f(x)\cos(3x)\ dx$?

(d) What is $\int_{-\pi}^{\pi} f(x)\sin(3x)\ dx$?

3. Prove the following statement: If $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$, then $b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\ dx$.

# Introduction to Differential Equations

1. Find all functions $f$ such that $f'$ is continuous and for all $x$

$$[f(x)]^2 = 100 + \int_0^x \left((f(t))^2 + (f'(t))^2\right)\ dt$$

2. Suppose that $f(x)$ is a solution to the initial value problem $\frac{dy}{dx} = 2x - y$, $y(1) = 5$.

   (a) If $f(a) = -4$ and $f'(a) = -2$, what is $a$?

   (b) Is $f$ increasing or decreasing at $x = 1$?

   (c) Find $f''(x)$.

   (d) If $f(4) = 2$, does $f$ have a critical point, and inflection point, or neither at $x = 4$?

3. Recall that we have already learned how to differentiate a power series. Use this to show that $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ is a solution to the initial value problem $\frac{d^2y}{dx^2} = -y, \quad y(0) = 1$.

4. Let $f$ be a function such that

   - $f(0) = 1$
   - $f'(0) = 1$
   - $f(a+b) = f(a)f(b)$ for all $a$ and $b$

   Prove that $f'(x) = f(x)$. Consequently, as we've seen in class, $f(x)$ must equal $e^x$.

# Separation of Variables

1. Suppose you forgot the Product Rule for differentiation, and instead thought $\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}(f(x))\right)\left(\frac{d}{dx}(g(x))\right)$. You get lucky, and get the correct answer for $\frac{d}{dx}(f(x)g(x))$ when $f(x) = e^{x^2}$. What was $g(x)$?

# Slope Fields and Euler's Method

1. Recall that an equilibrium solution to a differential equation is a solution that is constant. Some equilibrium solutions can be classified as either **stable** or **unstable**. If solutions curves tend toward an equilibrium solution, we call that a stable equilibrium. If solution curves tend away from an equilibrium solution, we call that an unstable equilibrium. Consider the differential equation:

$$\frac{dy}{dx} = 0.5y(y-4)(2+y)$$

   (a) What are the equilibrium solutions of this differential equation?

   (b) Sketch the slopefield.

   (c) Classify each equilibrium solution as stable, unstable, or neither.

   (d) If $y(0) = 6$, what is $\lim_{x\to\infty} y(x)$?

   (e) If $y(0) = -1$, what is $\lim_{x\to\infty} y(x)$?

2. Consider the initial value problem $\frac{dy}{dt} = e^{y^3}, \quad y(0) = y_0$

   (a) Find $\frac{d^2y}{dt^2}$.

   (b) Using Euler's method with $n = 10$ steps to estimate $y(2)$, would you over or under estimate the true value of $y(2)$? Why?

   (c) Suppose you now use Euler's method with $n = 100$ steps in order to estimate $y(2)$. Would this approximation be greater than or less than the approximation discussed in (b)? Explain.

# Population Growth Models and Logistic Growth

1. The table below gives the percentage, $P$, of households with a VCR, as a function of $t$ in years.

| $t$ | 1978 | 1979 | 1980 | 1981 | 1982 | 1983 | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 | 1991 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P$ | 0.3 | 0.5 | 1.1 | 1.8 | 3.1 | 5.5 | 10.6 | 20.8 | 36.0 | 48.7 | 58 | 64.6 | 71.9 | 71.9 |

   (a) Explain why a logistic model is reasonable for this data.

   (b) Use the data to estimate the point of inflection of $P$. What limiting value does this point of inflection predict?

   (c) As it turns out, the best model for this data is

$$P(t) = \frac{75}{1 + 316.75e^{-0.699t}}$$

   What limiting value does this model predict?