Math 501 Homework #5, Fall 2023

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: noon on Thursday 30 November 2023

Exercises /60

1. Use the Euclidean algorithm to calculate $d = \gcd(69, 372)$ and express d as a linear combination of 69 and 372. /3
2. Prove that, in a Euclidean domain, if a divides bc and $ab \neq 0$ then $\frac{a}{\gcd(a,b)}$ divides c. Is the same necessarily true in an arbitrary PID? /3
3. Show that every localization of a PID is a PID. Exhibit a commutative domain R and a submonoid S of R such that R is not a PID but $R[S^{-1}]$ is a PID and not a field. /3
4. Prove that the field $\mathbb{Q}$ of rational numbers has no nontrivial automorphisms as a ring. /3
5. Show that $\langle x, y\rangle$ is not a principal ideal in $R[x, y]$ for any commutative ring R. /3
6. Prove that $\mathbb{Z}[x_1, x_2, x_3, \ldots]/\langle x_1x_2, x_3x_4, \ldots\rangle$ has infinitely many minimal prime ideals. /3
7. Describe all of the ideals of $\mathbb{F}[x]/\langle g\rangle$, where $\mathbb{F}$ is a field and $g \in \mathbb{F}[x]$ is any polynomial. /3
8. Let R be a commutative integral domain containing a field $\mathbb{F}$ as a subring. Prove that if R has finite dimension as a vector space over $\mathbb{F}$ then R is a field. /3
9. Fix a module M over a commutative ring R with a multiplicative submonoid S of R. Figure out how to define $M[S^{-1}]$, and show that it is a module over $R[S^{-1}]$. /3
10. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules over a commutative ring R with a submonoid S, show that $0 \to M'[S^{-1}] \to M[S^{-1}] \to M''[S^{-1}] \to 0$ is exact. /3
11. Each prime ideal $\mathfrak{p}$ in the commutative ring R yields a localization homomorphism $M \to M_{\mathfrak{p}} = M[(R \smallsetminus \mathfrak{p})^{-1}]$. Show that this homomorphism needn't be injective, but the natural map $M \to \prod_{\mathfrak{m}} M_{\mathfrak{m}}$ is injective, the product being over all maximal ideals $\mathfrak{m}$. /3
12. In the notation of the previous two exercises, prove that $0 \to M' \to M \to M'' \to 0$ is exact if and only if $0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$ is exact for all prime ideals $\mathfrak{p}$ of R. /3
13. If R is an integral domain, M is a torsion-free R-module, and $\mathfrak{p}$ is a prime ideal of R, prove that the localization homomorphism $M \to M_{\mathfrak{p}}$ is injective. /3
14. Fix a left R-module M. Set $N = \{m \in M \mid xm = 0 \text{ for some } 0 \neq x \in R\}$. What conditions on R guarantee that N is a submodule of M? Does commutativity help? /3
15. Prove that the *annihilator* $\operatorname{ann}(M) = \{x \in R \mid xm = 0 \text{ for all } m \in M\}$ of a left R-module M is a two-sided ideal of R. /3

16. Is the ideal $\langle x, y \rangle$ in the polynomial ring $\mathbb{F}[x, y]$ a free module over $\mathbb{F}[x, y]$? /3

17. If M is a left R-module and $e \in R$ is a *central idempotent* of R, meaning that $e^2 = e$ and $ex = xe$ for all $x \in R$, then prove that $M = eM \oplus (1 - e)M$. /3

18. Fix an ideal $I \subseteq R$. Prove or disprove: R/I is a nonzero free R-module implies $I = 0$. /3

19. Prove that if every finitely generated module over a commutative ring R is free, then R is either a field or the zero ring. /3

20. Formulate and prove a version of the Chinese Remainder Theorem for a module M and ideals $I_1, \ldots, I_n$. What is the weakest "coprimality" condition you can think of to place on the ideals, given M? /3