

# Math 501 Homework #3, Fall 2023

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: noon on Thursday 12 October 2023

## EXERCISES

/60

1. The dihedral group of symmetries of a square acts on the set consisting of the diagonals of the square. What is the stabilizer of one of the diagonals? /3
2. What is the stabilizer of the first standard basis vector under the left action of  $GL_n(\mathbb{F})$  on the column vectors of size  $n$ , where  $\mathbb{F}$  is a field? /3
3. Let  $S = \mathbb{F}^{m \times n}$  be the  $m \times n$  matrices over a field  $\mathbb{F}$ . Describe the orbit decomposition of  $S$  under the action of  $G = GL_m(\mathbb{F}) \times GL_n(\mathbb{F})$  by  $(A, B) \cdot M = AMB^{-1}$ . /3
4. Describe all ways in which  $S_3$  can operate on a set of four elements. /3
5. For groups  $K \leq H \leq G$ , prove  $[G : K] = [G : H][H : K]$  without assuming  $G$  is finite. /3
6. Show by example that if  $H$  and  $K$  are finite index subgroups of  $G$ , then  $[H : H \cap K]$  need not divide  $[G : K]$ . /3
7. The dihedral group of symmetries of a square acts on the set of vertices; is that action faithful? What about the action on the diagonals? /3
8. A group  $G$  acts on a set of five elements with two orbits, one of size 2 and one of size 3. What are the possibilities for  $G$ ? /3
9. The octahedral group  $O$  acting by rotation on the cube. What is the stabilizer of a body diagonal? /3
10. Prove that the icosahedral group has a subgroup of order 10. /3
11. Determine the class equation of the dihedral group  $D_n$ . /3
12. Classify the groups of order 8. /3
13. Prove that every group of order 35 is cyclic. /3
14. Prove that the tetrahedral group is isomorphic to the alternating group  $A_4$ . /3
15. If  $p$  is the smallest prime dividing  $|G|$  and  $H \trianglelefteq G$  has order  $p$ , then  $H \leq Z(G)$ . /3
16. Prove that no group of order  $p^2q$  is simple if  $p$  and  $q$  are prime. /3
17. Find a Sylow  $p$ -subgroup of  $GL_2(\mathbb{F}_p)$ . /3
18. If  $p^e \parallel |G|$  with  $p$  prime, show that  $G$  has a subgroup of order  $p^r$  for all  $r \leq e$ . /3
19. Prove that the only simple groups of order  $< 60$  have prime order. /3
20. Show that there are at most five isomorphism types of groups of order 20. /3