Math 501 Homework #2, Fall 2023

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: noon on Thursday 21 September 2023

Exercises /66

1. Determine the automorphism groups of the integers $\mathbb{Z}$, the symmetric group S_3, and the cyclic group C_{10}. /3
2. Find all subgroups of S_3 and determine which are normal. /3
3. Given two homomorphisms φ and ψ from a group G to G', let $H \subseteq G$ be the subset where φ and ψ agree: $H = \{x \in G \mid \varphi(x) = \psi(x)\}$. Is H a subgroup of G? /3
4. Prove that the center of any group is a normal subgroup. /3
5. If $\varphi : G \twoheadrightarrow G'$ is a surjective homomorphism and $N \trianglelefteq G$ is a normal subgroup, prove that the image $\varphi(N) \trianglelefteq G'$ is also a normal subgroup. /3
6. Is the intersection $R \cap R'$ of two equivalence relations in $S \times S$ an equivalence relation on S? Is the union? /3
7. Prove that every group whose order is a power of a prime p contains an element of order p. /3
8. Let $\mathbb{F}$ be a field and W the solution set in $\mathbb{F}^n$ of a system of homogeneous linear equations $A\mathbf{x} = \mathbf{0}$. Show that the solution set of any inhomogeneous system $A\mathbf{x} = \mathbf{b}$ is a coset of W. /3
9. Prove that every index 2 subgroup is normal. Exhibit a non-normal index 3 subgroup. /3
10. Classify all groups of order 6. Hint: is there an element of order 6? Of order 3 but not of order 6? Or no element of order 3? /3
11. If G and G' are finite groups whose orders are relatively prime, prove that there is a unique homomorphism $G \to G'$. /3
12. Fix subgroups H and K of a group G. Prove that the intersection $xH \cap yK$ of cosets is either empty or else is a coset of $H \cap K$. Conclude that if H and K have finite index in G then so does $H \cap K$. /3
13. Prove that a group of order 30 can have at most seven subgroups of order 5. /3
14. Fix a surjective group homomorphism $\varphi : G \to G'$ with kernel K. Show that the set of subgroups of G containing K and the set of all subgroups of G' are in bijection via the map $H \mapsto \varphi(H)$. If $H \trianglelefteq G$, must it be that $\varphi(H) \trianglelefteq G'$? /3
15. Is the symmetric group S_3 a direct product of nontrivial groups? /3

16. Prove that the product of two infinite cyclic groups is not cyclic. Is the same true without the word "infinite"? /3

17. Fix a group G whose order is $|G| = ab$. Suppose that G has subgroups H and K with orders $|H| = a$ and $|K| = b$. Assume that $|H \cap K| = 1$. Prove that $HK = G$. Is G isomorphic to the product group $H \times K$? /3

18. Suppose that a group G has a partition P with the property that for any pair of blocks A and B of the partition, the product AB is contained entirely within a block of P. Let N be the block that contains the identity e of G. Prove that $N \trianglelefteq G$ and that P is the partition of G into the set of cosets of N. /3

19. Let $H = \{\pm 1, \pm i\} \subset \mathbb{C}^\times$, the subgroup of fourth roots of unity. Describe the cosets of H in $\mathbb{C}^\times$ explicitly (geometrically), and prove that $\mathbb{C}^\times / H \cong \mathbb{C}^\times$. /3

20. Fix a group G. Let $N = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$ be the subgroup of G generated by the *commutators* of pairs of elements of G. Prove that N is normal and the quotient G/N is abelian. Moreover, show that any homomorphism $G \to G'$ to an abelian group G' contains N in its kernel. /3

21. Assume that both H and K are normal subgroups of a group G and that $|H \cap K| = 1$. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. Hint: prove that $xyx^{-1}y^{-1} \in H \cap K$. /3

22. Find a nonabelian group G and a proper normal subgroup N such that G/N is abelian. /3