Math 501 Homework #1, Fall 2023

Instructor: Ezra Miller

Solutions by: ...your name...

Collaborators: ...list those with whom you worked on this assignment...

Due: noon on Thursday 7 September 2023

Exercises /48

1. Prove that set of invertible elements in any monoid is a group. /3
2. Solve for y given that $xyz^{-1}w = 1$ in a group. /3
3. Assume that the equation $xyz = 1$ holds in a group G. Does it follow that $yzx = 1$? How about $yxz = 1$? /3
4. Fix elements a and b in a group G. Show that the equation $ax = b$ has a unique solution in G. /3
5. Determine the elements of the cyclic group generated by the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ explicitly. /3
6. Let a and b be elements of a group G. Assume that a has order 5 and that $a^3b = ba^3$. Prove that $ab = ba$. /3
7. Prove that a nonempty subset H of a group G is a subgroup if for all $x, y \in H$ the element xy^{-1} lies in H. /3
8. An n^{th} *root of unity* is a complex number z such that $z^n = 1$. Prove that the n^{th} roots of unity form a cyclic subgroup of $\mathbb{C}^\times$ of order n. More generally, show that every finite subgroup of the multiplicative group of any field is cyclic. /3
9. Let H be the subgroup generated by two elements a and b of a group G. Prove that if $ab = ba$ then H is abelian. /3
10. Describe all groups that contain no proper subgroup. Describe all groups that contain no proper nontrivial subgroup. /3
11. Let G be a cyclic group of order n, and let r be an integer dividing n. Prove that G contains exactly one subgroup of order r. /3
12. Let G be a cyclic group of order 6. How many of its elements generate G? How about if G has order 5, 8, or 10? And the general case of order n? /3
13. Prove that a group in which every element except the identity has order 2 is abelian. /3
14. How many elements of order 2 does the symmetric group S_4 have? What about S_5 or S_6? /3
15. Prove that the set of elements of finite order in an abelian group is a subgroup. Find a group whose elements of finite order do not constitute a subgroup. /3
16. Let M be a finite monoid that satisfies the cancellation law. Prove that M is a group. /3