Pf of Prop 1: $M_{tor} = \ker(M \to M_{\langle 0\rangle})$, where $M_{\mathfrak{p}} = S^{-1}M$ for $S = R\setminus\mathfrak{p}$.

Suffices: $E = \mathrm{im}(M \to M_{\langle 0\rangle})$ is free, since then $0 \to M_{tor} \to M \to E \to 0$ splits.

Why is rank F well defined? Because $F \xrightarrow{\sim} M/M_{tor}$!

Will prove $E \subseteq$ f.g. free submodule of $M_{\langle 0\rangle}$, so Thm p. 47 applies.

$M = \langle m_1,\dots,m_k\rangle \Rightarrow M_{\langle 0\rangle} = \mathrm{span}_{R_{\langle 0\rangle}}\{\frac{m_1}{1},\dots,\frac{m_k}{1}\}$ over $R_{\langle 0\rangle} = K(R) =$ field.

$E = \langle \frac{m_1}{1},\dots,\frac{m_k}{1}\rangle$ over R.

Choose basis $x_1,\dots,x_n$ for $M_{\langle 0\rangle}$.

Let $a \in R$ be $\prod$(all denominators in coeffs on $x_1,\dots,x_n$ in $\frac{m_1}{1},\dots,\frac{m_k}{1}$).

Then $E \subseteq R\frac{x_1}{a} \oplus \cdots \oplus R\frac{x_n}{a}$. □

Ex: rank $M = n$.

Lemma: $M(p) = \ker(M \to M[p^{-1}])$. □

Lemma: If $M = M_{tor}$ then $M(p) \hookrightarrow M$ induces an isomorphism $M(p)_{\langle p\rangle} \xrightarrow{\sim} M_{\langle p\rangle}$.

Pf: Need $\twoheadrightarrow$ by HW5 #9b. But

$\frac{x}{a} \in M_{\langle p\rangle} \Rightarrow p^n\frac{x}{a} = 0$ since $M_{\langle p\rangle}$ is torsion and $R_{\langle p\rangle}$ has just one prime $\neq \langle 0\rangle$ local + PID

$\Rightarrow bp^nx = 0$ for some $b \notin \langle p\rangle$

$\Rightarrow bx \in M(p)$

$\Rightarrow \frac{x}{a} = \frac{bx}{ab} \in M(p)_{\langle p\rangle}$. □

Lemma: $p^nN = 0 \Rightarrow N \xrightarrow{\sim} N_{\langle p\rangle}$.

Pf: $p^nN = 0 \Rightarrow N$ is a module over $R/\langle p^n\rangle$. Write $R \twoheadrightarrow R/\langle p^n\rangle$, $a \mapsto \bar{a}$.

Then $\bar{r} \in (R/\langle p^n\rangle)^*$ $\forall$ $r \notin \langle p\rangle$, since $\langle r, p^n\rangle = 1$.

Hence, if $S = R\setminus\langle p\rangle$ then $N_{\langle p\rangle} = S^{-1}N = S^{-1}RN = S^{-1}R/\langle p^n\rangle N = R/\langle p^n\rangle N = N$. □

Pf of Prop 2: $M(p) \neq 0 \Rightarrow \exists\ 0 \neq x_p \in M(p)$ with $px_p = 0$. Thus

$\langle x_p \mid M(p)\neq 0\rangle$ f.g. by Cor p. 47, so $\underbrace{M(p) = 0 \text{ for almost all } p}_{\bigoplus_p M(p) \cong \prod_p M(p).}$

This is crucial; like finiteness of rank, it is a key place where f.g. is used

Lemmas $\Rightarrow M(p) \to M_{\langle p\rangle} = M(p)_{\langle p\rangle} = M(p) \Rightarrow M(p) \xrightarrow{\sim} M_{\langle p\rangle}$ $\forall p$.

But $M(q)_{\langle p\rangle} = 0$ $\forall \langle q\rangle \neq \langle p\rangle$. Thus

$M \to \prod_p M_{\langle p\rangle} = \prod_p M(p) = \bigoplus_p M(p)$ becomes $\cong$ locally at every $\langle p\rangle$. □

Nakayama's Lemma: Fix local ring A with maximal ideal $\mathfrak{p}$ and f.g. A-module N.

Then $N = \mathfrak{p}N \Rightarrow N = 0$.

Pf: $N = \langle x_1, \dots, x_n \rangle \Rightarrow x_1 = \sum_{i=1}^{n} a_i x_i$ with $a_i \in \mathfrak{p}$ $\forall i$

$$\Rightarrow \underbrace{(1-a_1)}_{\in A \setminus \mathfrak{p} = A^*} x_1 = \sum_{i=2}^{n} a_i x_i \quad \text{with } a_i \in \mathfrak{p} \quad \forall i$$

$\Rightarrow x_1 \in \langle x_2, \dots, x_n \rangle$. Done by induction ($n = 0$ trivial). □

Nakayama's Lemma: $(A, \mathfrak{p})$ local ring with f.g. A-module M. Write

$$M \to M/\mathfrak{p}M$$
$$x \mapsto \bar{x}$$

Assume $M/\mathfrak{p}M = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$. Then $M = \langle x_1, \dots, x_n \rangle$.

Pf: Apply previous to $N = M/\langle x_1, \dots, x_n \rangle$, so $\mathfrak{p}N = \mathfrak{p}M + \langle x_1, \dots, x_n \rangle$

N is a set of cosets; so is $\mathfrak{p}N$

$0 = \mathfrak{p}M + \langle x_1, \dots, x_n \rangle / M + \langle x_1, \dots, x_n \rangle$ $\quad = M + \langle x_1, \dots, x_n \rangle$

again, a set of cosets

$= N$. □

E.g. $x_1, \dots, x_n$ basis of $F \Leftrightarrow \bar{x}_1, \dots, \bar{x}_n$ basis of $F/\mathfrak{p}F$.

Prop 3: Assume R local PID with $\langle p \rangle$ maximal and M f.g. with $p^e M = 0$.

If $0 \to K \to F \to M \to 0$ is exact with $F \cong R^n$,

then F has a basis $f_1, \dots, f_n$ such that

$p^{e_1} f_1, \dots, p^{e_n} f_n$ is a basis of K for some $e_1, \dots, e_n \in \mathbb{N}$.

Pf: $F/pF \twoheadrightarrow M/pM$.

F has basis $B = B' \cup B''$ with $B'' \leftrightarrow$ basis $\bar{B}''$ for M/pM.

Using B'', assume $F/pF \xrightarrow{\sim} M/pM$, so $pF \supseteq K$ because

$$\begin{tikzcd} K \arrow[r] \arrow[rr, bend left, "0"] & F \arrow[r] \arrow[rr, bend right, "\ker = pF"'] & M \arrow[r] & M/pM \end{tikzcd}$$

$e = 1$: Any basis of F will do, since $F/pF \xrightarrow{\sim} M = M/\underbrace{pM}_{=0}$.

$e \geq 2$: $pF/K = pM$ is killed by p^{e-1}.

Induction $\Rightarrow$ choose basis $g_1, \dots, g_n$ of pF with

$p^{e_1 - 1} g_1, \dots, p^{e_n - 1} g_n$ basis of K.

$g_i \in pF \Rightarrow g_i = pf_i$ for some (unique) $f_i \in F$, so $p^{e_i} f_i = p^{e_i - 1} g_i$.

Nakayama $\Rightarrow f_1, \dots, f_n$ basis of F. □

course evals