

25.

Structure thm for modules over PIDThm: Fix f.g. nonzero module  $M / \text{PID } R$ . $\exists!$  proper ideals  $\langle q_1 \rangle \subseteq \dots \subseteq \langle q_n \rangle$  Def: the invariants of  $M$ with  $M \cong R/\langle q_1 \rangle \oplus \dots \oplus R/\langle q_n \rangle$ .E.g.  $R = \mathbb{Z} \Rightarrow M \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{r = \text{rank } M} \oplus \underbrace{\mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_k\mathbb{Z}}_{\bigoplus_p \mathbb{Z}/p^{e_i(p)}\mathbb{Z}} \quad \text{with } n_k | \dots | n_1$  $e_1(p) \geq \dots \geq e_k(p) \quad \forall \text{ primes } p, \text{ and almost all } 0$ 

$$\begin{aligned} \text{e.g. } \mathbb{Z}^3 &\oplus \boxed{\mathbb{Z}/32\mathbb{Z}} \oplus \boxed{\mathbb{Z}/32\mathbb{Z}} \oplus \boxed{\mathbb{Z}/4\mathbb{Z}} \oplus \boxed{\mathbb{Z}/2\mathbb{Z}} \oplus \boxed{\mathbb{Z}/2\mathbb{Z}} \oplus \boxed{\mathbb{Z}/2\mathbb{Z}} \\ &\oplus \boxed{\mathbb{Z}/81\mathbb{Z}} \oplus \boxed{\mathbb{Z}/27\mathbb{Z}} \\ &\oplus \boxed{\mathbb{Z}/625\mathbb{Z}} \oplus \boxed{\mathbb{Z}/25\mathbb{Z}} \oplus \boxed{\mathbb{Z}/5\mathbb{Z}} \oplus \boxed{\mathbb{Z}/5\mathbb{Z}} \end{aligned}$$

Fundamental thm of f.g. abelian groups

$$\mathbb{Z}^3 \oplus \mathbb{Z}/\langle 162000 \rangle \oplus \mathbb{Z}/\langle 2160 \rangle \oplus \mathbb{Z}/\langle 20 \rangle \oplus \mathbb{Z}/\langle 10 \rangle \oplus \mathbb{Z}/\langle 2 \rangle \oplus \mathbb{Z}/\langle 2 \rangle$$

E.g.  $R = \mathbb{k}[x] \Rightarrow$  module  $V$  is a vector space  $/ \mathbb{k}$ with  $x \cdot v = T v$  for some  $T: V \rightarrow V$ Thm  $\Rightarrow V \cong \mathbb{k}[x]^r \oplus (\dim < \infty)$ 

$\uparrow$   
easy, so assume  $\dim V < \infty$  this is Math 221!

$$\text{Thm} \Rightarrow V \cong \bigoplus_p \mathbb{k}[x]/\langle p^{e_i(p)} \rangle \oplus \dots \oplus \bigoplus_p \mathbb{k}[x]/\langle p^{e_k(p)} \rangle$$

 $e_1(p) \geq \dots \geq e_k(p) \quad \forall \text{ primes } p, \text{ and almost all } 0$ Q. What is  $\{p\}$  the set of all prime elements up to units?

A. all irreducible polynomials up to units

Q. What if  $\mathbb{k} = \mathbb{C}$ ?A.  $p = x - \alpha$  for  $\alpha \in \mathbb{C}$ Q. For fixed  $p = x - \alpha$ , what is  $\mathbb{k}[x]/\langle p^{e_i(p)} \rangle \oplus \dots \oplus \mathbb{k}[x]/\langle p^{e_k(p)} \rangle$ ?i.e.  $\mathbb{k}[x]/\langle (x - \alpha)^e \rangle \oplus \dots \oplus \mathbb{k}[x]/\langle (x - \alpha)^e \rangle$ ?Q. What is  $\mathbb{k}[x]/\langle (x - \alpha)^e \rangle$ ?A. cyclic, with minimal polynomial  $p(x) = (x - \alpha)^e$ = a Jordan block of size  $e$  for eigenvalue  $\alpha$  by Lec. 23.Conclusion: over  $\mathbb{k}[x]$ , Thm  $\Leftrightarrow$  Jordan form thm

course evals

## Pf overview

Recall: M/commutative domain R has torsion submodule

$$M_{\text{tor}} = \{m \in M \mid rm = 0 \text{ for some } r \in R \setminus \{0\}\}.$$

Fix M f.g./PID R.

Prop 1:  $M = M_{\text{tor}} \oplus F$  with

- F free
- $\text{rank } M \stackrel{\text{def}}{=} \text{rank } F < \infty$  well defined.

Prop 2:  $M = M_{\text{tor}} \Rightarrow M = \bigoplus_p M(p)$  where

- $p^e M(p) = 0$  for some  $e \in \mathbb{N}$
- $M(p) = 0$  for almost all p. immediate from  $\bigoplus$ , since M f.g.

$M(p) = \{m \in M \mid p^e m = 0 \text{ for } e \gg 0\}$ . analogue of Sylow p-subgroup!

Prop 3:  $M = M(p) \Rightarrow M \cong R/\langle p^{e_1} \rangle \oplus \dots \oplus R/\langle p^{e_k} \rangle$ .

$\exists$ : Props  $\Rightarrow M \cong R^l \oplus \bigoplus_p R/\langle p^{e_{l+1}(p)} \rangle \oplus \dots \oplus R/\langle p^{e_{l+k}(p)} \rangle$  with  $e_{l+1}(p) \geq \dots \geq e_{l+k}(p)$ .

Set  $q_1, \dots, q_l = 0$

and  $q_i = \prod_p p^{e_i(p)}$  for  $i > l$ , where  $e_i(p) = 0$  for  $i > l+k$ .

Then  $\dots | q_n | \dots | q_1$  by construction, so  $\langle q_1 \rangle \subseteq \dots \subseteq \langle q_n \rangle \subseteq \dots$   $n = \max_{e_i(p) \neq 0} i$

$\cdot \bigoplus_p R/\langle p^{e_i(p)} \rangle \cong R/\langle q_i \rangle$  by CRT. ✓

!:  $\#\{i \mid q_i = 0\} = \text{rank } M$ . Henceforth assume  $M = M_{\text{tor}}$ .

Suppose  $q_1, \dots, q_n$  satisfy thm.

Uniqueness is trivial when  $q_1 \dots q_n = p$  is prime:  $n=1$  and  $q_1 = p$ .

If  $q_1 \dots q_n = pa$  with  $a \notin R^*$

then  $\#\{i \mid p \mid q_i\} = \dim_{R/\langle p \rangle} M/pM$  since

$$pR/\langle q \rangle \cong \begin{cases} R/\langle q \rangle & \text{if } p \nmid q \\ R/\langle q/p \rangle & \text{if } p \mid q. \end{cases}$$

Induction on # prime factors of  $q_1 \dots q_n \Rightarrow$  uniqueness for  $pM$ :

$\text{ord}_p\left(\frac{q_i}{\gcd(p, q_i)}\right)$  is well defined.

Hence  $\text{ord}_p(q_i)$  is well defined, too: add 1 or not, as appropriate. □