

23.

Products, sums, and free modules

Def: A product of objects $\{M_i\}_{i \in I}$ in a category is

- an object P with
- morphisms $P \xrightarrow{\pi_i} M_i \forall i \in I$ satisfying
- a universal property: $N \xrightarrow{\psi_i} M_i \forall i \in I \Rightarrow \exists! N \rightarrow P$ making $\begin{array}{ccc} N & \xrightarrow{\exists! \quad \pi_i} & P \\ & \searrow \psi_i & \downarrow \pi_i \\ & & M_i \end{array}$ commute.

"A product is terminal in the category of objects with morphisms to all M_i "

E.g. In $A\text{-MOD}$, $P = \prod_{i \in I} M_i$. Same notation in any category, if P exists.

Def: A coproduct of objects $\{M_i\}_{i \in I}$ in a category is

an object Q that is universal (initial) in the category of objects with morphisms from all M_i .

Thus $M_i \rightarrow N \forall i \in I \Rightarrow \exists! Q \rightarrow N$ making $\begin{array}{ccc} M_i & \longrightarrow & Q \\ & \searrow & \downarrow \\ & & N \end{array}$ commute.

E.g. In $A\text{-MOD}$, $Q = \bigoplus_{i \in I} M_i$, the direct sum.

Discuss infinite cases: $\bigoplus_{i \in I} M_i \hookrightarrow \prod_{i \in I} M_i$, with $\cong \Leftrightarrow |I| < \infty$

Proofs of E.g.s: $A = \mathbb{Z} \checkmark$

$A = \text{arbitrary}$: check universal maps are A -linear. \square

E.g. $\{M_i\}_{i \in I}$ family of submodules of M induces

$\bigoplus_{i \in I} M_i \rightarrow M$. Def: direct sum decomposition if \cong .

Def: $x \in M$ generates cyclic submodule $Ax \subseteq M$.

A family $\{x_\lambda \mid \lambda \in \Lambda\}$ of elements of M is linearly independent over A if

$$\bigoplus_{\lambda \in \Lambda} A \xrightarrow{\cong} \bigoplus_{\lambda \in \Lambda} Ax_\lambda \xrightarrow{\varphi} M. \quad \cong : x_\lambda \text{ nzd} \quad \hookrightarrow : M \text{ decomposes as} \\ \oplus \text{ cyclic submods}$$

Lemma: $\Leftrightarrow \left(\sum_{\lambda \in \Lambda} a_\lambda x_\lambda = 0 \Rightarrow a_\lambda = 0 \forall \lambda \right)$

Pf: $\ker \varphi = \{\text{linear dependence relations on the } x_\lambda\}$
 $\stackrel{\text{def}}{=} \{\text{syzgyies on the } x_\lambda\}$. \square

Note: x_λ could equal x_λ for $\lambda \neq \lambda'$.

Q. When is $M = M_1 \oplus \dots \oplus M_n$ for $M_1, \dots, M_n \subseteq M$?

A. $M = M_1 + \dots + M_n$ and $M_i \cap \sum_{j \neq i} M_j = 0 \forall i$.

Def: M is free if it has a generating linearly independent subset.

basis

S any set \rightsquigarrow free module with basis S is $\bigoplus_{s \in S} A \stackrel{\text{def}}{=} A^{\oplus S}$
 \cong finitely supported functions $S \rightarrow A$

Thm: $\{x_\lambda\}_{\lambda \in \Lambda}$ basis of M and

$\{y_\lambda\}_{\lambda \in \Lambda}$ any elements of N

$\Rightarrow \exists!$ homomorphism $\varphi: M \rightarrow N$ with

$$\varphi(x_\lambda) = y_\lambda \quad \forall \lambda \in \Lambda.$$

Pf: $M = A$ with basis $\{1\}$: cyclic submodule $\Rightarrow Ax_\lambda \rightarrow N \quad \forall \lambda$

General case: universal property of \bigoplus $\Rightarrow \bigoplus_\lambda Ax_\lambda \rightarrow N$. \square

Def: For a module M , a (free) presentation is a morphism

$F_1 \rightarrow F_0$ with cokernel $F_0 \twoheadrightarrow M$ (with both F_i free).

E.g. $T: V \rightarrow V$ vector space/ \mathbb{k} $\dim < \infty$ V is a $\mathbb{k}[x]$ -module via $x \cdot v = Tv$

$V = \langle v \rangle$ cyclic $\Leftrightarrow V = \text{span}_{\mathbb{k}}\{v, Tv, \dots, T^{d-1}v\}$ for $d = \dim V$

$$\Leftrightarrow \mathbb{k}[x] \twoheadrightarrow V$$

$$\begin{array}{rcl} 1 & \mapsto & v \\ x & \mapsto & ?Tv \end{array} \quad \text{ker } = ? \text{ minimal polynomial } p(x) \text{ of } T$$

$$\Leftrightarrow \mathbb{k}[x] \xrightarrow{p(x)} \mathbb{k}[x]$$

$$f \mapsto p(x)f \quad \text{is a presentation of } V.$$

Def: An exact sequence $0 \rightarrow K \xrightarrow{\psi} M \xrightarrow{\varphi} N \rightarrow 0$ splits if

it has a section σ with $\varphi\sigma = \text{id}_N$.

Note: Sequence is short exact $\Rightarrow M = K + \sigma(N)$ by #5 on midterm 2.

But $K \cap \sigma(N) = 0$ since $\sigma(N) \hookrightarrow N$.

Hence split $\Leftrightarrow M = K \oplus \sigma(N)$.

often written: $M = K \oplus N$. $\text{but wrong: needs } \cong, \text{ not } =$

Cor: N free \Rightarrow sequence splits.

Pf: If $\{x_\lambda\}_{\lambda \in \Lambda}$ in M maps to basis $\{y_\lambda\}_{\lambda \in \Lambda}$ of N , then

use Thm to construct $\sigma: y_\lambda \mapsto x_\lambda \quad \forall \lambda \in \Lambda$. \square