

Modules

Fix a ring A .

Def: M is a left module over A , or a left A -module if M is

- an abelian group with

- a left action of (A, \cdot) that is distributive: $\begin{cases} \cdot (x+y)m = xm + ym \\ \cdot x(m+n) = xm + xn \end{cases}$

Ex: $-m = -1 \cdot m$, $x(-m) = -(xm)$, $0m = 0$

Assume "module" = "left module"

Def: $N \subseteq M$ is a submodule if $AN \subseteq N$.

E.g. • A is a free A -module with basis $1 \in A$.

- $I \subseteq A$ submodule \Leftrightarrow left ideal

- A/I cyclic A -module generated by $\bar{1} \in A/I$

- \mathbb{Z} -module = ? abelian group

- \mathbb{k} field $\Rightarrow \mathbb{k}$ -module = ? vector space/ \mathbb{k}

- $V = \text{vector space}/\mathbb{k}$ and $T: V \rightarrow V \Rightarrow V$ is a $\mathbb{k}[x]$ -module via $x \cdot v = T^v$

- $A = R^{n \times n} = n \times n$ matrices/ring $R \Rightarrow R^n = \{\boxed{\quad}\}$ left ideal $f(x) \cdot v = f(T)v$
and $R_{\text{row}}^n = \{\boxed{\quad}\}$ right ideal pick which column

- set S and A -module $M \Rightarrow \text{maps}(S \rightarrow M)$ is an A -module via $(af)(s) = a(\underbrace{f(s)}_{\in M})$

Def: M is generated by $\{m_\lambda \mid \lambda \in \Lambda\} \subseteq M$ if $m \in M \Rightarrow m = \sum_{\lambda \in \Lambda} x_\lambda m_\lambda$

for $\{x_\lambda \mid \lambda \in \Lambda\} \subseteq A$ almost all 0.

linear combination

E.g.: $\mathbb{Z} \times \mathbb{Z}$ is generated by $\{[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]\}$ but also $\{[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]\}$.

Lemma: $N \subseteq M$ submodule \Rightarrow group M/N is naturally a module.

Pf: $xm \equiv xm' \pmod{N}$ if $m - m' \in N$ because then $xm - xm' = x(m - m') \in N$. \square

E.g. $\underbrace{M}_{\mathbb{Z} \times \mathbb{Z}} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow im = N \cong \mathbb{Z}$. What is M/N ?

$\mathbb{Z} \times \mathbb{Z} \leftarrow \mathbb{Z}$ Use \circlearrowleft : $N = 3\mathbb{Z}[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}] \times 0 \subseteq \mathbb{Z}[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}] \times \mathbb{Z}[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}] \Rightarrow M/N \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}$.

Def: Fix module M over a commutative domain R .

The torsion submodule is $M_{\text{tor}} = \{m \in M \mid rm = 0 \text{ for some } r \in R \setminus \{0\}\}$

E.g.: $(M/N)_{\text{tor}} \cong \mathbb{Z}/3\mathbb{Z}$.

Def: $I \subseteq A$ left ideal $\Rightarrow IM = \{ \text{linear combinations of elements of } M \text{ with coefficients in } I \}$. (4)

Ex: IM is a submodule. $I(JM) = (IJ)M$ and $(I+J)M = IM + JM$.

Def: $\varphi: M \rightarrow M'$ is a module homomorphism if φ is a group homomorphism

$(M, +) \rightarrow (M', +)$ with $\varphi(xm) = x\varphi(m) \quad \forall x \in A \text{ and } m \in M$.
 \uparrow
 φ is A-linear

Note: $A\text{-Mod}$ is a category: objects M • $1_M: M \rightarrow M$ identity
morphisms $\varphi: M \rightarrow M'$ • $\varphi: M \rightarrow M'$ and $\psi: M' \rightarrow M'' \Rightarrow \psi \circ \varphi: M \rightarrow M''$

$A\text{-Mod}$ has initial object: $\mathbb{Z}0$ • associative and $1_M \circ \varphi = \varphi = \varphi \circ 1_{M'}$.
terminal object: $\mathbb{Z}0$

Ex: $\ker(M \rightarrow M') \subseteq M$ submodule

$\text{im}(M \rightarrow M') \subseteq M'$ submodule

Def: $\text{coker}(M \rightarrow M') = M'/\text{im}(M \rightarrow M')$ quotient module

Isomorphism theorems

1. $N, N' \subseteq M$ submodules $\Rightarrow N+N' \subseteq M$ submodule and

$N/N \cap N' \xrightarrow{\sim} (N+N')/N'$. Pf: $N \twoheadrightarrow (N+N')/N'$ has kernel $N \cap N'$; apply universal property.

2. $M \supseteq M' \supseteq M'' \Rightarrow M/M' \xrightarrow{\sim} (M/M'')/(M'/M'')$. Pf: similar.

3. homomorphism $\varphi: M \rightarrow M'$ and $N' \subseteq M' \Rightarrow \varphi^{-1}(N') \subseteq M$ submodule and

$$M/\varphi^{-1}(N') \hookrightarrow M/N'.$$

Def: The sequence $M \xrightarrow{\psi} M' \xrightarrow{\varphi} M''$ is exact at M if $\text{im } \psi = \ker \varphi$.

If $\varphi \circ \psi = 0$ then the sequence has homology $\ker \varphi / \text{im } \psi$.

Lemma: $N \subseteq M \Rightarrow$ short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$.

Pf: Discuss: $0 \rightarrow N \rightarrow M$ exact $\Leftrightarrow N \hookrightarrow M$ monomorphism (or embedding)

$M \rightarrow M' \rightarrow 0$ exact $\Leftrightarrow M \twoheadrightarrow M'$ epimorphism.

$N \rightarrow M \rightarrow M/N$ exact at M . \square

Def: Fix a commutative ring R . An R -algebra is a ring A with a ring homomorphism $R \xrightarrow{f} A$ such that $\text{im } f \subseteq \text{center}(A)$.

Lemma: $\Rightarrow A$ is an R -module. \square