Localization

Fix a commutative ring R

Def: $S \subseteq R$ is a __multiplicative subset__ if S is a submonoid of $(R, \cdot)$.

- $1 \in S$ and
- $xy \in S \quad \forall \, x, y \in S$

The __ring of fractions__ is

$$S^{-1} R = R[S^{-1}] = R \times S / \sim$$

where the class of (a, s) is denoted $\frac{a}{s}$

and $\frac{a}{s} = \frac{a'}{s'}$ if $\underbrace{\exists \, t \in S \text{ with } t(s'a - sa') = 0.}$

$s'a - sa'$ is annihilated by something that's supposed to be a unit.

Note: $\frac{a}{s} = \frac{a'}{s'}$ and $\frac{a'}{s'} = \frac{a''}{s''} \Rightarrow s''t' \, t(s'a - sa') = 0 = -st \, t'(s''a' - s'a'')$

$\Rightarrow s't't(s''a - sa'') = 0$

$\Rightarrow \sim$ is transitive   symmetric and reflexive: easy

Prop: $R[S^{-1}]$ is a ring with $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$ and $\frac{a}{s} + \frac{b}{t} = \frac{ta + bs}{st}$.

Pf: e.g. $\frac{a}{s} = \frac{a'}{s'} \Rightarrow u(s'a - sa') = 0$

$\Rightarrow u(s'tab - a'bst) = 0$

$\Rightarrow \frac{ab}{st} = \frac{a'b}{s't}$. $\square$

Cor: $R \to R[S^{-1}]$ is a ring homomorphism.

$a \mapsto \frac{a}{1}$

next page

Q. $R = \mathbb{R}[x,y]/\langle xy \rangle \Rightarrow \ker(R \to R[x^{-1}]) = ? \, \langle y \rangle$

stuff annihilated by something that's supposed to be a unit!

E.g. 1. $S = R^* \Rightarrow R[S^{-1}] = R$.   We'll see why shortly, with the universal property

2. R integral domain and $S = R \setminus \{0\}$

$\Rightarrow R[S^{-1}] = K(R) = $ __fraction field__ of R   not "quotient field": $R/\mathfrak{m}$ vs. $K(R)$

e.g. $R = \mathbb{Z} \Rightarrow K(R) = \mathbb{Q}$

$R = \mathbb{k}[x_1, \ldots, x_n] \Rightarrow K(R) = \mathbb{k}(x_1, \ldots, x_n)$ __field of rational functions__ in $x_1, \ldots, x_n$ over $\mathbb{k}$

3. $\mathfrak{p} \subseteq R$ prime ideal and $S = R \setminus \mathfrak{p}$

$\Rightarrow S^{-1}R \overset{\text{def}}{=} R_{\mathfrak{p}}$, the __localization__ of R at $\mathfrak{p}$ #2: $\mathfrak{p} = 0$

$R_{\mathfrak{p}}$ is a __local ring__: it has a unique maximal ideal   Pf: Exercise

4. $S = \{1, t, t^2, \ldots\}$

$$\Rightarrow S^{-1}R = R[S^{-1}] \overset{def}{=} R_t$$

$\ker(R \to R_t) = ?$ $\quad \frac{a}{t} = \frac{0}{1} \Leftrightarrow ua = 0$ for some $u \in S$

$$\Leftrightarrow t^d a = 0 \text{ for some } d \in \mathbb{N}$$

e.g. $R = \Bbbk[x,y]/\langle xy \rangle \Rightarrow R = \Bbbk\{1, x, x^2, \ldots, y, y^2, \ldots\}$ $\quad$ basis as vector space $/\Bbbk$

$x^d f(x,y) = 0$ for some $d \in \mathbb{N} \Leftrightarrow xy \mid x^d f(x,y)$

$$\Leftrightarrow y \mid f(x,y)$$

$\overset{/\langle y \rangle}{\Rightarrow} \ker(R \to R[x^{-1}]) = \langle y \rangle$

<u>Prop</u>: {prime ideals of $S^{-1}R$} $\leftrightarrow$ {prime ideals $\mathfrak{p} \subseteq R$ with $\mathfrak{p} \cap S = \emptyset$}

$$\mathfrak{p}S^{-1}R \leftarrow\!\shortmid \mathfrak{p}$$

the point: $I \subseteq R$ remains a proper ideal $\Leftrightarrow I$ has no element that becomes a unit

<u>Pf</u>: $\qquad \mathfrak{p} \mapsto \{a \in R \mid \frac{a}{1} \in \mathfrak{p}\}.$ The rest: Exercise. $\square$

<u>Prop</u>: Let $\mathcal{C}$ = category of ring homomorphisms $R \xrightarrow{f} A$ such that

$f(s) \in A^*$ $\forall s \in S$.

morphism: $R \xrightarrow{f} A$, $\downarrow$, A'

Then $R \to S^{-1}R$ is universally repelling in $\mathcal{C}$:

$$R \xrightarrow{f} A, \quad \nearrow S^{-1}R \downarrow \exists! f_*$$

<u>Pf</u>: Let $f: R \to A$ be an object in $\mathcal{C}$.

Define $f_*: S^{-1}R \to A$ by $\frac{a}{s} \mapsto f(a)f(s)^{-1}$ $\quad$ uses $f(s) \in A^*$

Then $\frac{a}{s} = \frac{a'}{s'} \Rightarrow t(s'a - sa') = 0$ for some $t \in S$

$$\Rightarrow \left(f(t)(f(s')f(a) - f(s)f(a'))= 0\right) \cdot f(t)^{-1} f(s')^{-1} f(s)^{-1}$$

$$\Rightarrow f(a)f(s)^{-1} - f(a')f(s')^{-1} = 0,$$

so f_* is well defined.

Exercise: f_* · is a ring homomorphism and

· makes the diagram commute. $\quad$ — this is by def.

f_* is unique because it is determined by where it sends R. $\square$

<u>Cor</u>: $R[(R^*)^{-1}] = R.$

<u>Pf</u>: Any ring homomorphism $R \to A$ factors through $R \xrightarrow{id} R$,

so $R \xrightarrow{id} R$ satisfies the universal property. $\square$