

17.

RingsDef: A is a ring if it has

Anneau

$$+: A \times A \rightarrow A$$

$$\cdot : A \times A \rightarrow A$$

with  $\cdot (A, +)$  abelian group $\cdot (A, \cdot)$  monoid

$$\text{distributivity: } x(y+z) = xy + xz$$

$$\text{and } (y+z)x = yx + zx \quad \forall x, y, z \in A$$

E.g. Why is  $- \cdot - = +$  ?

$$1 + (-1) = 0 \Rightarrow x + (-1)x = 0x \quad 0 + 0 = 0 \quad \text{by def of identity}$$

$$\Rightarrow (-1)x = -x$$

$$\Rightarrow 0x + 0x = 0x$$

$$\Rightarrow -(xy) = (-x)y$$

$$\Rightarrow 0x = 0$$

$$\Rightarrow -(xy) + (-x)(-y) = (-x)y + (-x)(-y)$$

$$= (-x)(y + (-y))$$

$$= 0$$

$$\Rightarrow xy = (-x)(-y)$$

Def:  $u \in A$  is a unit if  $\exists v \in A$  with  $uv = 1$   
 and  $w \in A$  with  $wu = 1$

$$\Rightarrow w(uv) = (wu)v$$

$$\Rightarrow w = v.$$

 $A^* = \text{unit group of } A$ 

E.g. 1.  $A = \mathbb{k}[x] = \{a_0 + a_1x + \dots + a_dx^d \mid d \in \mathbb{N} \text{ and } a_0, \dots, a_d \in \mathbb{k}\} \Rightarrow A^* = \mathbb{k}^*$

2.  $\mathbb{Z}^{\mathbb{N}} = (a_0, a_1, \dots) \Rightarrow R = \text{Hom}_{\text{Ab}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}})$

$= \{f: \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}^{\mathbb{N}} \mid f(\underline{a} + \underline{b}) = f(\underline{a}) + f(\underline{b})\}$  is a ring.

PF?

$T(a_0, a_1, \dots) = (0, a_0, a_1, \dots) \Rightarrow T$  has left inverse but not right.

3.  $A^* = A \setminus \{0\} \Rightarrow A$  is a division ring (skew field)

and  $xy = yx \quad \forall x, y \in A \Rightarrow A$  is a field

A is a commutative ring note: not "abelian"

4.  $\mathbb{Z}$  is a commutative integral domain ( $\mathbb{Z}$  is entire):  $\neq 0$  and  $ab = 0 \Rightarrow a = 0$  or  $b = 0$  (34)

5.  $\mathbb{Z}/6\mathbb{Z}$  is commutative with zero divisors:  $\bar{2} \cdot \bar{3} = \bar{6} = 0$  in  $\mathbb{Z}/6\mathbb{Z}$

6. monoid algebras: monoid  $G$  and ring  $A$

$$\Rightarrow R = A[G] = \left\{ \sum_{g \in G} a_g g \mid \text{almost all } a_g = 0 \right\} \quad \text{e.g. } R = \mathbb{k}[x] = \mathbb{k}[\mathbb{N}]$$

$$\alpha = \sum_{g \in G} a_g g \quad \text{and} \quad \beta = \sum_{g \in G} b_g g \Rightarrow \alpha \beta = \sum_{g \in G} \sum_{h \in G} a_g b_h g h$$

convolution product

$$= \sum_{x \in G} \left( \sum_{gh=x} a_g b_h \right) x$$

This is just the usual product on polynomials

7. function rings  $X$  set (top. space manifold analytic space complex manifold algebraic variety)

yes if  $|X| \geq 2$   $R$  ring  $\mathbb{R}$   $\mathbb{R}$   $\mathbb{R}$   $\mathbb{C}$   $\mathbb{k}$  Körper

$$A = \{f: X \rightarrow R\} \quad \text{continuous} \quad C^\infty \quad \text{analytic} \quad \mathbb{C}\text{-analytic} \quad \text{algebraic}$$

$$f+g: x \mapsto f(x)+g(x) \quad \text{yes} \quad \text{yes} \quad \text{no} \longrightarrow$$

$$fg: x \mapsto f(x)g(x)$$

commutative? yes if  $R$  is

zero divisors?

8. matrix rings  $M_n(R) = n \times n$  matrices with entries in ring  $R$

$$\mathbb{k} \text{ field} \Rightarrow M_n(\mathbb{k})^* = GL_n \mathbb{k} \quad (\det \neq 0)$$

commutative? only if  $n=1$  and  $R$  is commutative

zero divisors? yes unless  $n=1$  and  $R$  is an integral domain

Def:  $f: A \rightarrow B$  is a ring homomorphism if

$$(A, +) \rightarrow (B, +) \text{ is an abelian group homomorphism} \quad f(a+a') = f(a)+f(a'), \quad f(0) = 0$$

$$(A, \cdot) \rightarrow (B, \cdot) \text{ is a monoid morphism} \quad f(aa') = f(a)f(a') \quad f(1) = 1$$

Lemma:  $\exists!$  ring homomorphism  $\mathbb{Z} \rightarrow A$  for any ring  $A$ .

Pf:  $1 \mapsto 1$ .  $\square$

$$\ker = n\mathbb{Z}. \quad n = p \text{ prime} \Rightarrow A \text{ has characteristic } p \Rightarrow A \supseteq \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

0

0

$\mathbb{Z}$

Def:  $A \subseteq B$  is a subring if the inclusion is a homomorphism.

E.g.  $A \subseteq B$  subring and  $S \subseteq B$  subset  $\Rightarrow A[S] = \left\{ \sum_{\text{finite}} a_{s_1} \cdots a_{s_n} s_1 \cdots s_n \right\} \subseteq B$  subring