

# Simplicity of $A_n$

Thm:  $A_n$  is simple if  $n \geq 5$ .

Pf: Induction on  $n$ .

$n = 5$ :  $A_5 \cong I$  is simple by Cor p. 22

Assume  $n \geq 6$ . Let  $H \trianglelefteq A_n$ .

Set  $G_i = G_{A_n}(i)$ , so  $G_i \cong A_{n-1} \trianglelefteq A_n$  simple  $\forall i = 1, \dots, n$ .

$1 \neq \pi \in H$  with  $\pi(i) = i \Rightarrow |H \cap G_i| \geq 1$

$\Rightarrow H \geq G_i$  since  $H \trianglelefteq A_n$ .

But  $\sigma \in A_n \Rightarrow \sigma G_i \sigma^{-1} = G_{\sigma(i)}$

$\Rightarrow H \geq G_j \forall j$ , again since  $H \trianglelefteq A_n$ .

Every  $\rho \in A_n$  is a product of an even # of transpositions

= " " some # of pairs of transpositions

each lies in  $G_i$  for some  $i$ , as  $n \geq 4$

Hence  $\langle G_1, \dots, G_n \rangle = A_n$ , so  $H = A_n$ .

So assume  $\pi(i) \neq i \forall \pi \in H \setminus \{1\}$  and  $\forall i \in \{1, \dots, n\}$ .

Then  $\pi_1(i) = \pi_2(i) \Rightarrow \pi_1 = \pi_2$  in  $H$  because  $\Downarrow$

$$\pi_1^{-1} \pi_2(i) = i$$

Suppose  $\pi$  has a  $\geq 3$ -cycle  $(a_1, a_2, a_3, \dots)$  in its cycle decomposition.

Fix  $\sigma \in A_n$  with  $\sigma(a_1) = a_1$

and  $\sigma(a_2) = a_2$

but  $\sigma(a_3) \neq a_3$ , possible because  $n \geq 5$ :  $(a_3 a_4 a_5)$ .

Then  $\pi' = \sigma \pi \sigma^{-1}$  has the  $\geq 3$ -cycle  $(a_1 a_2 \sigma(a_3) \dots)$  in its cycle decomposition.

(\*)  $\pi'(a_1) = \pi(a_1) = a_2 \Rightarrow \pi \notin H$ , else  $\pi' \in H$  since  $H \trianglelefteq A_n$ , so  $\pi^{-1} \pi'(a_1) = a_1$ .

Now suppose  $\pi = (a_1 a_2)(a_3 a_4)(a_5 a_6) \dots$  = product of disjoint transpositions.

Set  $\sigma = (a_1 a_2)(a_3 a_5) \in A_n$ . Then

$$\pi' = \sigma \pi \sigma^{-1} = (a_2 a_1)(a_5 a_4)(a_6 a_3) \dots$$

$\Rightarrow (*)$  again  $\Rightarrow \pi \notin H$

$\Rightarrow |H| = 1$ .  $\square$

# Composition series and related concepts

Def: A composition series of a group  $G$  is a chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_{l-1} \trianglelefteq G_l = G \quad \text{such that } \underbrace{G_i/G_{i-1}}_{\text{composition factor } \bar{G}_i} \text{ is simple } \forall i = 1, \dots, l.$$

E.g.  $1 \trianglelefteq A_n \trianglelefteq S_n \quad \forall n \geq 5 \Rightarrow \bar{G}_1 = A_n \text{ and } \bar{G}_2 = C_2$

$1 \trianglelefteq C_2 \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4 \xrightarrow{\text{go this way}} \bar{G}_i = C_2, C_2, C_3, C_2$

Jordan-Hölder Thm: All composition series of finite  $G$  yield the same isomorphism classes and multiplicities of  $\bar{G}_1, \dots, \bar{G}_l$  up to permutation.

Def:  $G$  is solvable if  $|\bar{G}_i|$  is prime  $\forall i$ . "solvable" has roots in Galois theory

E.g.  $S_n$  is solvable if  $n \leq 4$  but not if  $n \geq 5$ . How to tell if  $G$  is solvable?

Prop: Set  $G' = [G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$  commutator subgroup and

$$G = G^0 \geq G^1 \geq \dots \geq G^i \geq \dots \quad \text{with } G^{i+1} = (G^i)', \text{ the derived series of } G.$$

Then finite  $G$  is solvable  $\Leftrightarrow G^m = \{1\}$  for some  $m$ .

Pf: Lemma:  $G/N$  is abelian  $\Leftrightarrow N \geq [G, G]$ .

Pf:  $\begin{matrix} \updownarrow & & \updownarrow \\ xyx^{-1}y^{-1} \in N & \forall x, y \in G \end{matrix} \quad \square$

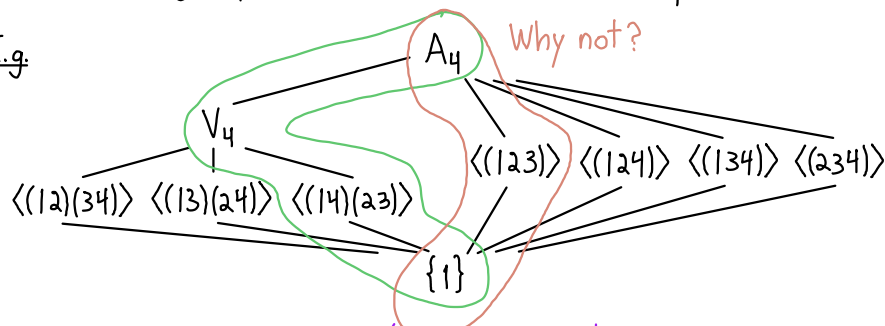
$\Leftarrow$ : Refine the derived series.

$\Rightarrow$ : Need  $G^i \neq \{1\} \Rightarrow (G^i)' \neq G^i$ . Jordan-Hölder  $\Rightarrow G^i$  has an abelian quotient  $C_p = G^i/H$   
 $\Rightarrow G^i \not\geq H \geq (G^i)'$ .  $\square$

Q. What do all composition series look like, together in  $G$ ?

A. The subgroup lattice of  $G$  is  $\Lambda(G) = \text{poset of all subgroups of } G$ .

E.g.



Q.  $N \trianglelefteq G \Rightarrow \Lambda(G/N) = ?$  A.  $\{\bar{H} \leq \bar{G} \mid N \leq H \leq G\}$ , where  $-$  means  $/N$   
 and  $\bar{H} \trianglelefteq \bar{G} \Leftrightarrow H \trianglelefteq G$

Ex:  $N \leq H \leq G$  with  $N \trianglelefteq G \Rightarrow G/H \xleftarrow{\bar{G}/\bar{H}} \bar{G}/\bar{H} \cong \text{if } H \trianglelefteq G$  one of the isomorphism theorems