15. Free groups

Thm: Fix a set X. There is a free group F on X with the following universal property:

$$\text{for any group } G, \ \operatorname{maps}(X \to G) = \operatorname{Hom}(F, G). \quad = \{\text{homomorphisms } F \to G\}$$

Intuition: F is generated by X with "no relations"

Compare: B = basis for vector space V $\Rightarrow \operatorname{maps}(B \to W) = \operatorname{Hom}(V, W)$ $\forall$ v.s. W

Def: A word on X of length n is an element $w \in X^n$. e.g. $X = \{a, \dots, z\}$ $w = \text{aardvark}$

$W = \bigcup_{n=0}^{\infty} X^n$ is the free semigroup on X: $v \cdot w = vw$ $\quad$ aardvark $\cdot$ syzygy = aardvarksyzygy

$X^{-1} = \{x^{-1} \mid x \in X\}$ $\qquad X' = X^{-1} \cup X$ $\qquad$ Set $(x^{-1})^{-1} = x$ for $x^{-1} \in X^{-1}$.

Def: The equivalence relation $\sim$ on W' = free semigroup on X' is the transitive closure of

$$v \sim w \text{ if } v = v_1 y y^{-1} v_2 \text{ and } w = v_1 v_2 \text{ for some } y \in X'. \quad \text{determined by}$$

$w \in W'$ is reduced if no string yy^{-1} appears in w.

Lemma: $w \sim w_0$ for some reduced w_0. $\square$

e.g. $ba\cancel{b}\cancel{b^{-1}}a^{-1}c^{-1}ca = babb^{-1}a^{-1}\cancel{c^{-1}}\cancel{c}a$

$baa^{-1}\cancel{c^{-1}}\cancel{c}a \qquad babb^{-1}\cancel{a^{-1}}\cancel{a}$

$b\cancel{a}\cancel{a^{-1}}a \qquad ba\cancel{b}\cancel{b^{-1}}$

$ba \qquad ba$

Prop: w reduces to unique reduced w_0 by a sequence of cancellations.

Pf: Induct on length $\ell(w)$. $\quad$ Notation: $w \rightsquigarrow v$ if w reduces to v.

$\ell(w) = 0 \Rightarrow w = 1$ (empty word) is reduced.

$\ell(w) > 0$: Assume $w \rightsquigarrow w_0 \neq w$ (else done) starts $w_1 \cancel{x}\cancel{x^{-1}} w_2$.

Suffices: $w \rightsquigarrow w_0' \Rightarrow w_1 w_2 \rightsquigarrow w_0'$, since already $w_1 w_2 \rightsquigarrow w_0$.

In w_0, x or x' must go. • Both simultaneously: may as well do them first.

• One first $\Rightarrow$ at $\cdots \cancel{x^{-1}}\cancel{x}x^{-1} \cdots$ or $\cdots x \cancel{x^{-1}}\cancel{x} \cdots$

$\Rightarrow$ same as if xx^{-1} had been canceled first. $\square$

Cor: w_0 in Lemma is unique, and $w \rightsquigarrow w_0$.

Pf: $w \sim w_0' \Rightarrow$
$$\begin{array}{ccccccccc} & v_r & & & & v_1 & & v_0 & \\ \swarrow & & \searrow & \cdots & \swarrow & & \searrow\swarrow & & \searrow \\ w & & & & & w_1 & & & w_0' \end{array}$$
and may as well assume w_i reduced $\forall\, i < r$ since
$$\begin{array}{ccc} & v_i & \\ \swarrow & & \searrow \\ w_{i+1} & & w_i \\ \Downarrow & & \Downarrow \\ w_{i+1}' & = & w_i' \end{array}$$

But then $= \forall i$ and $w_0 = w_r'$ by Prop. $\square$

Prop: $v \sim v'$ and $w \sim w' \Rightarrow vw \sim v'w'$.

Pf: $vw \rightsquigarrow v_0 w \rightsquigarrow v_0 w_0 \rightsquigarrow (v_0 w_0)_0 \leftsquigarrow v_0 w_0 \leftsquigarrow v_0 w' \leftsquigarrow v'w'$. $\square$

Cor: $F_X = W'/\sim$ is a group. $\square$ $\quad$ E.g. $X = \{a\} \Rightarrow F_X \cong \cancel{?}\ C_\infty$

$F_X \times F_X \to F_X$ ✓
assoc. since concatenation is
$1 \cdot w = w \cdot 1$ by def.
$(xy \cdots z)^{-1} = z^{-1} \cdots y^{-1} x^{-1}$

Pf of Thm: Given $f: X \to G$ and $w = x_1^{\pm 1} \cdots x_n^{\pm 1} \in W'$, define $\varphi: F_X \to G$ by

$\varphi(w) = f(x_1)^{\pm 1} \cdots f(x_n)^{\pm 1}$. Then $w \sim w' \Rightarrow \varphi(w) = \varphi(w')$ since G is a group.

φ is a homomorphism by construction. $\square$ $\quad \varphi(vw) = \varphi(x_1^{\pm 1} \cdots x_n^{\pm 1} y_1^{\pm 1} \cdots y_m^{\pm 1}) = f(x_1)^{\pm 1} \cdots f(y_m)^{\pm 1} = \varphi(v)\varphi(w)$.

Generators and relations

Notes: • $X \subseteq G$ generating set $\Leftrightarrow$ $F_X \twoheadrightarrow G$.

• $X \subseteq G \Rightarrow \mathrm{im}(F_X \to G) = \langle X \rangle \leqslant G$ subgroup generated by X

Def: If $\langle X \rangle = G$ then $\ker(F_X \twoheadrightarrow G) = N \leqslant F_X$ consists of the relations on X.

So $N = \{$equivalence classes of words on X' whose product in G is $1\}$.

E.g. $S_3 = G \supseteq X = \{\overset{x}{(12)}, \overset{y}{(23)}\} \Rightarrow N = \{x^2, y^2, xyxyxy, xy^2x, xy^2x^{-1}, \dots\}$ Who wants to write them all?

Def: If $G = \langle X \rangle$ then $R \subseteq W'$ is a set of defining relations if

$N = \ker(F_X \twoheadrightarrow G)$ is the smallest normal subgroup of F_X containing R.

Notes: • R need not generate N. Why? $F_X \twoheadrightarrow G$ group homomorphism $\Rightarrow N \trianglelefteq F_X$ and $G \cong F_X/N$.

• F_X has non-normal subgroups if $|X| \geqslant 2$ since $\exists\, G = \langle x, y \rangle \geqslant H$ with $H \ntrianglelefteq G$. e.g. $\langle (12) \rangle \leqslant S_3$

Thm (Universal property of quotient groups):

Fix $N \trianglelefteq G$ and $\overline{G} = G/N$ with $\pi : G \to \overline{G}$, $a \mapsto aN$.

$\varphi : G \to G'$ and $N \leqslant \ker\varphi \Rightarrow \exists!\ \overline{\varphi} : \overline{G} \to G'$ with $\varphi = \overline{\varphi} \circ \pi$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G' \\ \overline{G} \arrow[ur, "\exists!\ \overline{\varphi}"'] & \end{tikzcd}$$

Pf: Exercise. (!) Main point: partition of G by G/N refines partition by $G/\ker\varphi$ — it's the map of sets that matters.

E.g. $D_n = \langle x, y \mid x^n, y^2, xyxy \rangle$ defining relations R: $D_n \cong F_{\{x,y\}}/N$ $x^n, y^2, xyxy \in N \triangleleft F_{\{x,y\}}$ minimal

Pf: $\left.\begin{array}{l} x = \text{rotation by } 2\pi/n \\ y = \text{(any) reflection} \end{array}\right\}$ $x^n = 1$, $y^2 = 1$, and $(xy)^2 = 1$

$D_n = \langle x, y \rangle \Rightarrow F_{\{x,y\}} \xrightarrow{\varphi} D_n$ surjective.

$\left.\begin{array}{l} \ker\varphi \triangleleft F_{\{x,y\}} \\ \ker\varphi \supseteq R \end{array}\right\}$ $\Rightarrow \ker\varphi \geqslant N$

$\Rightarrow F_{\{x,y\}}/N \overset{\overline{\varphi}}{\twoheadrightarrow} D_n$

Need: $\overline{\varphi}$ injective.   Enough: $|F_{\{x,y\}}/N| \leqslant 2n$.

Ex: Using R, every word in $x^{\pm1}, y^{\pm1}$ can be put into the form

$\#\left(x^i y^j \text{ with } i \in \{0,\dots,n-1\} \text{ and } j \in \{0,1\}.\right) = 2n.$ □

Q. Given $R \subseteq F_X$ and $v, w \in X'$, is $v = w$ in $\langle X \mid R \rangle$?

A. word problem: $\exists$ no algorithm with bounded time complexity!

Q. $\langle X \mid xyx^{-1}y^{-1}$ for $x, y \in X \rangle = ?$ free abelian group on X $= \mathbb{Z}^X$.