

Free groups

Thm: Fix a set X . There is a free group F on X with the following universal property:

for any group G , $\text{maps}(X \rightarrow G) = \text{Hom}(F, G)$. = {homomorphisms $F \rightarrow G$ }

Intuition: F is generated by X with "no relations"

Compare: $B = \text{basis for vector space } V$
 $\Rightarrow \text{maps}(B \rightarrow W) = \text{Hom}(V, W) \quad \forall v.s. W$

Def: A word on X of length n is an element $w \in X^n$.

e.g. $X = \{a, \dots, z\}$ $w = \text{aardvark}$

$W = \bigcup_{n=0}^{\infty} X^n$ is the free semigroup on X : $v \cdot w = vw$ aardvark \cdot syzygy = aardvarksyzygy

$X^{-1} = \{x^{-1} \mid x \in X\}$ $X' = X^{-1} \cup X$ Set $(x^{-1})^{-1} = x$ for $x^{-1} \in X^{-1}$.

Def: The equivalence relation \sim on $W' = \text{free semigroup on } X'$ is the transitive closure of

$v \sim w$ if $v = v_1 y y^{-1} v_2$ and $w = v_1 v_2$ for some $y \in X'$. determined by

$w \in W'$ is reduced if no string yy^{-1} appears in w .

Lemma: $w \sim w_0$ for some reduced w_0 . \square

e.g. $\cancel{ba} \cancel{bb^{-1}} \cancel{a^{-1}c} \cancel{c^{-1}a} = \cancel{ba} \cancel{bb^{-1}} \cancel{a^{-1}c} \cancel{c^{-1}a}$

Prop: w reduces to unique reduced w_0 by a sequence of cancellations.

$\cancel{ba} \cancel{a^{-1}c} \cancel{c^{-1}a}$ ba bb^{-1} a^{-1}a

Pf: Induct on length $l(w)$.

Notation: $w \rightsquigarrow v$ if w reduces to v .

$\cancel{ba} \cancel{a^{-1}a}$ ba bb^{-1}

$l(w) = 0 \Rightarrow w = 1$ (empty word) is reduced.

ba ba

$l(w) > 0$: Assume $w \rightsquigarrow w_0 \neq w$ (else done) starts $w_1 \cancel{xx^{-1}} w_2$.

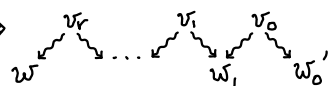
Suffices: $w \rightsquigarrow w_0' \Rightarrow w_1 w_2 \rightsquigarrow w_0'$, since already $w_1 w_2 \rightsquigarrow w_0$.

In w_0 , x or x^{-1} must go. • Both simultaneously: may as well do them first.

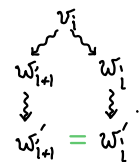
• One first \Rightarrow at $\dots \cancel{xx^{-1}} \dots$ or $\dots \cancel{xx^{-1}} \dots$

\Rightarrow same as if $\cancel{xx^{-1}}$ had been canceled first. \square

Cor: w_0 in Lemma is unique, and $w \rightsquigarrow w_0$.

Pf: $w \sim w_0' \Rightarrow$  and may as well assume w_i reduced $\forall i < r$ since

But then $= \forall i$ and $w_0 = w_r'$ by Prop. \square



Prop: $v \sim v'$ and $w \sim w' \Rightarrow vw \sim v'w'$.

Pf: $vw \rightsquigarrow v_0 w \rightsquigarrow v_0 w_0 \rightsquigarrow (v_0 w_0)_0 \rightsquigarrow v_0 w_0 \rightsquigarrow v_0 w' \rightsquigarrow v' w'$. \square

Cor: $F_X = W'/\sim$ is a group. \square E.g. $X = \{a\} \Rightarrow F_X \cong \mathbb{Z}$

$F_X \times F_X \rightarrow F_X$ \checkmark
 assoc. since concatenation is
 $1 \cdot w = w \cdot 1$ by def.
 $(xy \dots z)^{-1} = z^{-1} \dots y^{-1} x^{-1}$

Pf of Thm: Given $f: X \rightarrow G$ and $w = x_1^{\pm 1} \dots x_n^{\pm 1} \in W'$, define $\varphi: F_X \rightarrow G$ by

$\varphi(w) = f(x_1)^{\pm 1} \dots f(x_n)^{\pm 1}$. Then $w \sim w' \Rightarrow \varphi(w) = \varphi(w')$ since G is a group.

φ is a homomorphism by construction. \square $\varphi(vw) = \varphi(x_1^{\pm 1} \dots x_n^{\pm 1} y_1^{\pm 1} \dots y_m^{\pm 1}) = f(x_1)^{\pm 1} \dots f(y_m)^{\pm 1} = \varphi(v)\varphi(w)$.

- $X \subseteq G \Rightarrow \text{im}(F_X \rightarrow G) = \langle X \rangle \leq G$ subgroup generated by X

So $N = \{\text{equivalence classes of words on } X' \text{ whose product in } G \text{ is } 1\}.$

Who wants to write them all?

$N = \ker(F_X \twoheadrightarrow G)$ is the smallest normal subgroup of F_X containing R .

- F_X has non-normal subgroups if $|X| \geq 2$ since $\exists G = \langle x, y \rangle \geq H$ with $H \not\leq G$. e.g. $\langle (12) \rangle \leq S_3$

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \pi \downarrow & \nearrow \exists! \overline{\varphi} & \\ \overline{G} & & \end{array}$$
$$\varphi: G \rightarrow G' \text{ and } N \leq \ker \varphi \Rightarrow \exists! \bar{\varphi}: \bar{G} \rightarrow G' \text{ with } \varphi = \bar{\varphi} \circ \pi.$$

E.g. $D_n = \langle x, y \mid x^n, y^2, xyxy \rangle$ defining relations: $D_n \cong F_{\{x,y\}} / N$ $x^n, y^2, xyxy \in N \triangleleft F_{\{x,y\}}$
B minimal

$$D_n = \langle x, y \rangle \Rightarrow F_{\{x, y\}} \xrightarrow{\varphi} D_n \text{ surjective.}$$
$$\left. \begin{array}{l} \ker \varphi \triangleleft F_{\{x,y\}} \\ \ker \varphi \supseteq R \end{array} \right\} \Rightarrow \ker \varphi \supseteq N$$

$$\Rightarrow F_{\{x,y\}}/N \xrightarrow{\varphi} D_n$$

Need: $\bar{\varphi}$ injective. Enough: $|F_{\{x,y\}}/N| \leq 2n$.

$$\# (x^i y^j \text{ with } i \in \{0, \dots, n-1\} \text{ and } j \in \{0, 1\}) = 2n. \quad \square$$

A. word problem: \exists no algorithm with bounded time complexity!

Q. $\langle X | xyx^{-1}y^{-1} \text{ for } x, y \in X \rangle = ?$ free abelian group on $X = \mathbb{Z}^X$.