12.

Class eqn for I: $60 = 1+15+20+12+12$

$\lvert g\rvert$	#	
1	1	1
2	≥ 15	all I_e conjugate $\Rightarrow$ 15
3	$\geq 10\cdot 2 = 20$	" I_f " $\Rightarrow 20 = 20$ or $10+10$. But $I_f = \{1, h, h^2\} \Rightarrow h = \frac{2\pi}{3}$ rotation around pole(f) $\Rightarrow h^2 =$ " pole(−f) $\Rightarrow h \sim h^2$.
5	$\geq 6\cdot 4 = 24$	similarly, $I_v = \{1, g, g^2, g^3, g^4\} \Rightarrow 24 = 12+12$. □

x	$\lvert I_x\rvert$	# copies $\leq I$
face f	3	$20/2 = 10$
edge e	2	$30/2 = 15$
vertex v	5	$12/2 = 6$

all conjugate, since I acts transitively on faces, edges, and vertices

antipodal pairs

But $\overset{1}{1} + \overset{2}{15} + \overset{3}{20} + \overset{5}{24} = 60$, so that's all.

Conjugate elements have equal order, so class equation refines this sum.

Q. Why isn't this the class eqn?

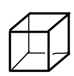
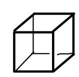
A. $24 \nmid 60$

Actions on subsets

$G \circlearrowright X \Rightarrow G \circlearrowright 2^X = \{\text{subsets of } X\}$

$U \subseteq X \Rightarrow gU = \{gu \mid u \in U\}$ same size as U

$\Rightarrow G \circlearrowright \binom{X}{d} = \{U \subseteq X \mid |U| = d\}$

E.g. G = octahedral group ↻  ⇒ G ↻ X = {vertices of  }

$\left|\binom{X}{2}\right| = \binom{8}{2} = 28$ pairs of vertices

3 orbits:
- $\mathcal{O}_1$ = pairs of vertices on an edge   # = 12
- $\mathcal{O}_2$ = " a face diagonal   # $= 6\cdot 2 = 12$
- $\mathcal{O}_3$ = " body diagonal   # $= 8/2 = 4$ $28 = 12+12+4$

Note again: $gU = U \Rightarrow g$ permutes U: not $gu = u$ but $gu \in U$

Lemma: If $H \circlearrowright X$ and $U \subseteq X$ then H stabilizes U ($H_U = H$) $\Leftrightarrow$ U is a union of H-orbits.

Pf: H stabilizes $U \Leftrightarrow \mathcal{O}_u = Hu \subseteq U \;\; \forall u \in U$. □

Prop: Let $G \circlearrowright G$ by left multiplication. Then $|G_U| \mid |U|$

Pf: Set $H = G_U$. Lemma $\Rightarrow U = \bigcup \underbrace{H\text{-orbits}}_{\text{right cosets } Hu}$.

$$\Rightarrow |U| = \sum_{H\text{-orbits } \mathcal{O}} |\mathcal{O}| = k|H|. \quad \square$$

Cor: $\gcd(|U|, |G|) = 1 \Rightarrow G_U = \{1\}$.

Pf: $|G_U| \mid |G|$ and $|G_U| \mid |U|$. □

E.g. $G \circlearrowright G$ by conjugation. $H \leq G \Rightarrow G_H$ is the normalizer

$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

subgroups conjugate to $H = \cancel{?}\ |G|/|N(H)|$ $\qquad |\mathcal{O}_H| \cdot |G_H| = |G|$

$= [G : N(H)]$.

Notes: • $H \trianglelefteq N_G(H)$ and $N(H)$ maximal with this property.

• $N_G(H) = G \Leftrightarrow H \trianglelefteq G$.

The Sylow theorems

Fix group G and prime p with $|G| = p^e m$ and $p \nmid m$. Assume $e \geq 1$.

First Sylow Thm: G has a subgroup of order p^e. Sylow p-subgroup (proofs next time)

Cor: G has an element of order p.

Pf: Let H be a Sylow p-subgroup and $1 \neq g \in H$.

$|g| = p^r$ for some $r \leq e \Rightarrow |g^{p^{r-1}}| = p$. $\square$

Second Sylow thm: Let $K \leq G$ with $p \mid |K|$ and $H \leq G$ a Sylow p-subgroup.

Then $(gHg^{-1}) \cap K$ is a Sylow p-subgroup of K for some $g \in G$.

Cor: 1. $K \leq G$ is a p-group $\Rightarrow K \leq$ some Sylow p-subgroup of G.

2. All Sylow p-subgroups of G are conjugate.

Pf: 1. Pick H as in Sylow 2. Then $K \leq gHg^{-1}$ by def.

But gHg^{-1} is a Sylow p-subgroup of G.

2. Part 1 + $K = gHg^{-1}$ if $|K| = |gHg^{-1}|$. $\square$

Third Sylow Thm: Let s = #Sylow p-subgroups of G, where $|G| = p^e m$.

Then $s \mid m$ and $s \equiv 1 \pmod p$.

E.g. $|G| = 15 \Rightarrow G \cong C_{15}$ $\qquad n = 15 = 3 \cdot 5$

$p = 3$, $m = 5$: $s \mid 5$ and $s \equiv 1 \pmod 3$.

$s \in \{1, 5\}$ $\qquad \Rightarrow s = 1 \Rightarrow$ Sylow 3-subgroup $H \triangleleft G$

$p = 5$, $m = 3$: $s \mid 3$ and $s \equiv 1 \pmod 5$. $\Rightarrow s = 1 \Rightarrow$ Sylow 5-subgroup $K \triangleleft G$

HW2 #21 $\Rightarrow$ G abelian

$\Rightarrow G \cong C_3 \times C_5 \cong C_{15}$.