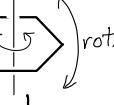


10.

Finite rotation groups

Thm: Every finite $G \leq SO_3$ is one of

- C_k cyclic group of rotations by $2\pi m/k$ about a fixed line for $m \in \mathbb{Z}$
- D_k dihedral = symmetries of  rot.
- T symmetries of tetrahedral group $|T| = 12$
- O symmetries of octahedral group $|O| = 24$
- I symmetries of icosahedral group $|I| = 60$

Pf: Set $n = |G|$. $1 \neq g \in G \Rightarrow g$ fixes unique line l

\Rightarrow " " " pair of points in unit sphere S^2
poles in S^2 of g 

G acts on $P = \bigcup_{1 \neq g \in G} \text{poles}(g)$: p is a pole of g and $a \in G$
 $\Rightarrow ap$ is a pole of aga^{-1}

$|P| = ?$ Dunno. Don't care, exactly.

Consider the multiset $M = \{ \text{poles}(g) \mid g \in G \text{ and } g \neq 1 \}$.

$$|M| = 2(n-1) = 2n-2$$

How many times does $p \in P$ appear in M ?

By def, $|G_p| - 1$. Set $r_p = |G_p|$. Then

$$|M| = 2n-2 = \sum_{p \in P} (r_p - 1).$$

Note: $G_p \cong C_{r_p}$ is generated by $\frac{2\pi}{r_p}$ rotation about $l = \text{span}(p)$.

$r_p \geq 2$ since $1 \neq g \in G_p$ if $p \in \text{poles}(g)$.

Let $\mathcal{O}_1, \mathcal{O}_2, \dots$ be the orbits of G in P .

Set $m_i = |\mathcal{O}_i|$.

Observe: $r_p = r_{p'} = \frac{n}{m_i}$ whenever $p, p' \in \mathcal{O}_i$ since $|G_p| \cdot |\mathcal{O}_p| = |G|$

$$\boxed{\quad} \Rightarrow 2n-2 = \sum_i m_i (r_i - 1)$$

$$\Rightarrow 2 - \frac{2}{n} = \sum_i \left(1 - \frac{1}{r_i}\right) \quad \text{famous formula}$$

$\nearrow < 2 \quad \searrow \geq \frac{1}{2}$

$$\Rightarrow \# \text{ orbits} \leq 3$$

$$\# \text{ orbits} = 1 : \underbrace{2 - \frac{2}{n}}_{\geq 1} = \underbrace{1 - \frac{1}{r}}_{< 1} \rightarrow *$$

$$\# \text{ orbits} = 2 : 2 - \frac{2}{n} = \left(1 - \frac{1}{r_1}\right) + \left(1 - \frac{1}{r_2}\right)$$

$$\frac{2}{n} = \left(\frac{1}{r_1}\right) + \left(\frac{1}{r_2}\right). \quad \text{But } r_i \leq n \text{ since } r_i | n$$

$$\geq \frac{1}{n} \geq \frac{1}{n}$$

$$\Rightarrow r_1 = r_2 = n$$

$$\Rightarrow |\mathcal{O}_1| = |\mathcal{O}_2| = 1$$

$$\Rightarrow P = \{\pm p\} \Rightarrow G = G_p \cong C_n.$$

$$\# \text{ orbits} = 3 : \frac{2}{n} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - 1. \quad \text{Assume } r_1 \geq r_2 \geq r_3.$$

$$\text{Then } r_1 = 2, \text{ else } \sum_{i=1}^3 \frac{1}{r_i} \leq 1.$$

Case 1: $r_1 = r_2 = 2$ and $r_3 = r \geq 2$

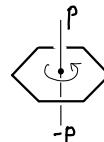
Case 2: $r_3 \geq 3$

$$1. \frac{2}{n} = \frac{1}{r_3} \Rightarrow n = 2r_3 \text{ and } |\mathcal{O}_3| = 2$$

$$\Rightarrow \mathcal{O}_3 = \{\pm p\} \text{ and } G_p = C_{r_3} \text{ about } l = \text{span}(p).$$

Every $g \in G$ fixes r -gon in l^\perp .

$$|G| = 2r \Rightarrow G = D_r$$



$$2. r = (2, r_2, r_3) \quad r_2 \geq 4 \Rightarrow \frac{1}{2} + \frac{1}{r_2} + \frac{1}{r_3} - 1 \leq 0 \rightarrow$$

$$r_2 \geq 3 \quad r_2 = 3 \text{ and } r_3 \geq 6 \Rightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{r_3} - 1 \leq 0 \rightarrow$$

$$\Rightarrow r \in \begin{cases} (2, 3, 3) \\ (2, 3, 4) \\ (2, 3, 5) \end{cases} \quad \frac{2}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - 1 = \frac{7}{6} - 1 = \frac{1}{6} \Rightarrow n = 12$$

$$n = 24$$

$$n = 60$$

E.g. $(2, 3, 5)$: $|\mathcal{O}_3| = \frac{60}{5} = 12$.

Let $p \in \mathcal{O}_3$ and $q \in \mathcal{O}_2$ a pole nearest to p .

$G_p \cap \mathcal{O}_2$ and $|G_p| = 5 \Rightarrow G_p \cdot q = \text{vertices of a regular pentagon}$

$\Rightarrow \#\{\text{poles nearest to } p\} = 5k$ for some k

Show $k = 1$ and conclude that the $|\mathcal{O}_3| = 12$ pentagons form dodecahedron

$\Rightarrow \mathcal{O}_3 = \text{vertices of icosahedron. } \square$