

9. Prop: G acts on $X \Leftrightarrow G \rightarrow S_X$ is a homomorphism,
 $g \mapsto \lambda_g$

where $\lambda_g: X \rightarrow X$
 $x \mapsto gx.$ \square

E.g. $D_3 \rightarrow S_3 = \text{permutations}(D_3/H)$

kernel = ? image = ?

$$\begin{aligned} & \cap \\ H \text{ since } g \cdot C_i = C_i \quad \forall i = 1, 2, 3 \Rightarrow gH = H \\ & \Rightarrow g \in H; \end{aligned}$$

but $[y \cdot C_2 = C_3, \text{ so } y \notin \ker(D_3 \rightarrow S_3).]$ Thus $\ker = \{1\}.$

or: $H \not\cong D_3$ since $xyx^{-1} = yx^2 \notin H,$ so $\ker \neq H$

$$\Rightarrow D_3 \cong S_3.$$

Prop: Fix a G -set X and $x \in X$ with stabilizer G_x and orbit $O_x.$

There is a natural bijection $G/H \xrightarrow{\Psi} O_x$ Every transitive group action
 $aH \mapsto ax$ is an action on cosets.

It satisfies $\Psi(gC) = g\Psi(C)$ for $g \in G$ and $C \in G/H.$

$$\begin{aligned} \text{Pf: Need } aH = bH \Rightarrow ax = bx. \text{ But } aH = bH &\Leftrightarrow a^{-1}b \in H \\ &\Leftrightarrow a^{-1}bx = x \end{aligned}$$

Ψ is surjective by construction.

injective by \square

Loose ends 1. $G \leqslant \text{Isom}(\mathbb{R}^2)$ finite $\Rightarrow G \cong C_n$ or $D_n.$

$$\begin{aligned} \text{Need } r \in G \text{ reflection} \Rightarrow G \cong D_n. \text{ Pf: } H = \{\text{rotations in } G\} \leqslant G \\ \Rightarrow H \cong C_n \text{ for some } n \\ \Rightarrow G \supseteq H \cup rH = D_n. \end{aligned}$$

But $g \in G$ reflection $\Rightarrow r^{-1}gr$ rotation
 $\text{act on } \Rightarrow g \in rH. \quad \square$

2. Prop: Let $G \subseteq X.$ Fix $x \in X$ and $x' \in O_x,$ say $x' = ax.$

Then $\{g \in G \mid gx = x'\} = aG_x$ and $G_{x'} = aG_xa^{-1}.$

$$\begin{array}{ccc} \text{Pf:} & | & | \\ x \mapsto x' & x \mapsto x & x' \mapsto x. \end{array} \quad \square$$

Counting

Recall: $|G| = |H| \cdot |G/H|$. Let $G \trianglelefteq X$.

Cor: $|G| = |\mathcal{O}_x| \cdot |\mathcal{O}_x|$ for $x \in X$.

Pf: $|G| = |H| [G:H]$ with $H = G_x$ plus bijection $G/H \rightarrow \mathcal{O}_x$. \square

Lemma: $|X| = \sum_{\text{orbits } O} |\mathcal{O}|$.

Pf: X is partitioned by its orbits. \square

E.g. G = orientation-preserving isometries of icosahedron I .

1. Q. $|G| = ?$

A. Every $g \in G$ is a rotation about centroid of I .

- G acts on $F = \{\text{faces of } I\}$. Let $f \in F$.

$$|\mathcal{O}_f| = |F| = ? \quad 20 \quad \text{because } G \text{ acts transitively}$$

$$|G_f| = ? \quad 3 \text{ rotations of } f = \triangle \Rightarrow |G| = 3 \cdot 20 = 60.$$

- G acts on $V = \{\text{vertices of } I\}$. Let $v \in V$.

$$|\mathcal{O}_v| = |V| = ? \quad 12 \quad \text{because } G \text{ acts transitively}$$

$$|G_v| = ? \quad 5 \text{ rotations fixing edges emanating from } v \Rightarrow |G| = 5 \cdot 12 = 60.$$

2. Q. How many edges does I have?

A. G acts on $E = \{\text{edges of } I\}$. Let $e \in E$.

$$|\mathcal{O}_e| = |E| = ? \quad 30 \quad \text{because } G \text{ acts transitively}$$

$$|G_e| = ? \quad 2 \text{ rotations fixing edge } e \Rightarrow |G| = 2 \cdot ? = 60.$$

3. $V \Rightarrow |V| = 12 = 1 + 1 + 5 + 5$
 $\text{vertex figures of } v \text{ and } -v$

$H = G_v = \text{stabilizer of } v \text{ for some (any) } v \in V$

$G_f = \text{stabilizer of } f \text{ for some (any) } f \in F \Rightarrow 12 = 3 + 3 + 3 + 3$
 H fixes no vertex

Prop: $H, K \leq G \Rightarrow [H : H \cap K] \leq [G : K]$.

Pf: Let $X = G/K$. Then $G \trianglelefteq X$. Set $x = 1K \in X$.

$$H_x = ? \quad H \cap K$$

$$[H : H \cap K] = |\mathcal{O}_x| \leq |X| = |G/K| = [G : K]. \quad \square$$

