

7. Groups of small order

G

$ G $	abelian	nonabelian
1	$\{e\}$	
2	$\mathbb{Z}/2\mathbb{Z}$	
3	$\mathbb{Z}/3\mathbb{Z}$	
4	$\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$a, b \in G \setminus \{e\}$ with $a^2 = e$ and $b^2 = e$ $\Rightarrow ab = ba$ since $\neq a, b, \text{ or } e$
5	$\mathbb{Z}/5\mathbb{Z}$	
6	$\mathbb{Z}/6\mathbb{Z}, \cancel{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}}$	S_3
7	$\mathbb{Z}/7\mathbb{Z}$	
8	$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^3$	 D_4, Q_8 ("quaternions")
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→ Def: The product $G \times G' = \{(g, g') \mid g \in G \text{ and } g' \in G'\}$ is a group under componentwise composition

$$(g_1, g'_1)(g_2, g'_2) = (g_1g_2, g'_1g'_2). \quad \text{Homomorphisms: } G \times G' \xrightarrow{\quad} G'$$

$\downarrow \quad \swarrow$
 projections

$$\begin{array}{ccc} G' & & g' \\ \downarrow & & \downarrow \\ G & \xrightarrow{\quad} & G \times G' \\ g & \mapsto & (g, e') \end{array}$$

E.g. $|G|=6$ nonabelian: $a, b \in G \quad ab \neq ba \quad |g| \in \{2, 3\} \quad \text{if } g \neq e$

$G = \{1, x, x^2, y, y^2, z\} \Rightarrow |z| = 2$, so G has an element of order 2.

But $a^2 = b^2 = (ab)^2 = e \Rightarrow abab = e \Rightarrow ab = ba$, so G has an element of order 3.

May as well assume $|a| = 2$ and $|b| = 3$, since then $ab = ba \Rightarrow G \cong \mathbb{Z}/6\mathbb{Z}$.

Then $G = \{1, a, b, b^2, ab, ba\} \cong S_3$ via $a \mapsto (12)$ and $b \mapsto (123)$.

Ex: $aba = ? \neq e, ab, ba, a$ (since $ab \neq e$), b (since $ab \neq ba$); check ↗

Quotient groups

Q. When is G/H a group? ... with $G \rightarrow G/H$ a homomorphism?

Thm: $\Leftrightarrow H$ is a normal subgroup, written $H \trianglelefteq G$.

Pf: \Rightarrow : Prop, p. ⑧: $G \xrightarrow{\varphi} G/H \Rightarrow H = \ker \varphi \trianglelefteq G$.

\Leftarrow : $(aH)(bH)$ is supposed to be a left coset of H , where

$$A, B \subseteq G \Rightarrow AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

"Every left coset is a right coset if H is normal":

Lemma: $H \trianglelefteq G \Leftrightarrow gH = Hg \quad \forall g \in G.$

$$\begin{aligned} \text{Pf: } gH &= \{gh \mid h \in H\} \\ &= \{g(g^{-1}hg) \mid h \in H\} \quad \text{since } g^{-1}Hg = H \\ &= \{hg \mid h \in H\} \\ &= Hg. \quad \square \end{aligned}$$

\Leftarrow also holds

$$\text{Now compute } (aH)(bH) = (a(Hb)H) = (a(bH)H) = abHH = abH.$$

$\frac{||}{ab} \quad \frac{||}{ab}$

Exercise: Let G be a group and S a set with a map $S \times S \rightarrow S$.
 $(s, t) \mapsto st$

If $\varphi: G \rightarrow S$ with $\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in G$

then S is a group.

Ex $\Rightarrow S = G/H$ proves Thm. \square

Cor: $N \trianglelefteq G \Leftrightarrow \exists$ homomorphism $\varphi: G \rightarrow G'$ with $\ker \varphi = N$.

Pf: \Rightarrow : Prop, p. 8

$\Leftarrow: G' = G/N. \quad \square$

$$\text{E.g. } n\mathbb{Z} \triangleleft \mathbb{Z} \quad n\mathbb{Z} = \ker(\mathbb{Z} \rightarrow C_n) \quad \mathbb{Z}/n\mathbb{Z} \cong C_n$$

General: What if $G \xrightarrow{\varphi} G'$ with $N = \ker \varphi$?

Is there a relation between G' and G/N ?

First Isomorphism Theorem: $G \xrightarrow{\varphi} G' \Rightarrow G/N \xrightarrow{\sim} G'$.

Pf: Essentially the Ex. \square

$$\text{E.g. } G = \mathbb{C}^*$$

$$N = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

$$G/N = ? \quad \text{cosets}$$

$$\begin{array}{ccc} \text{Diagram of concentric circles} & \cong & \mathbb{R}_+^* \\ |z|=4/3 & |z|=1/2 & |z|=2/3 \end{array}$$

How it's used: 1. $G/\ker \varphi \xrightarrow{\sim} \text{im } \varphi$

2. If $G \xrightarrow{\varphi} G'$ is any homomorphism and $N \trianglelefteq G$ with $N \leq \ker \varphi$, then φ induces a homomorphism $G/N \rightarrow G'$.