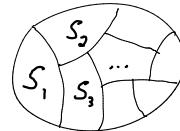


5.

Equivalence relations

3 ways to say same thing Fix a set S . block

1. partition of S : $S = \bigcup_{i \in I} S_i$ with $S_i \neq \emptyset$ and $S_i \cap S_j = \emptyset \quad \forall i \neq j$ disjoint union
index set (could be infinite)



E.g. 1. $\{1, 2, 3, 4, 5\} = \{1, 3\} \cup \{2, 5\} \cup \{4\}$ 2. $1 \sim 3 \quad 2 \sim 5$ 3. $\{1, 2, 3, 4, 5\} \rightarrow \{[3], [2], [4]\}$

• $\mathbb{Z} = \{\text{evens}\} \cup \{\text{odds}\}$

$a \sim b \Leftrightarrow 2 | a - b$

$\mathbb{Z} \rightarrow \{\bar{0}, \bar{1}\}$

evens $\mapsto \bar{0}$
odds $\mapsto \bar{1}$

2. equivalence relation \sim on S : for some pairs $a, b \in S$, $a \sim b$.

Must be • reflexive: $a \sim a$

Formally, $\sim \subseteq S \times S$

• symmetric: $a \sim b \Rightarrow b \sim a$

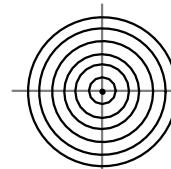
$\{(a, b) \mid a \sim b\}$

• transitive: $a \sim b$ and $b \sim c \Rightarrow a \sim c$ E.g. 1. + 2.

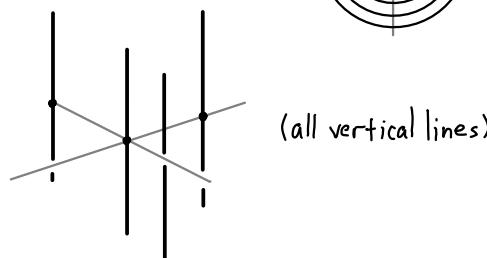
equivalence class of $a = [a] = \{x \in S \mid x \sim a\}$

3. fibers of a map $S \xrightarrow{\varphi} T$: the fiber over $t \in T$ is $\varphi^{-1}(t) = \{s \in S \mid \varphi(s) = t\}$.

E.g. $\mathbb{C} \rightarrow \mathbb{R}$
 $z \mapsto |z|$ \Rightarrow fibers:



E.g. $\mathbb{R}^3 \rightarrow \mathbb{R}^2$
 $(x, y, z) \mapsto (x, y)$ \Rightarrow fibers:



E.g. 3.

3 \Rightarrow 1: $S = \bigcup_{t \in T} \varphi^{-1}(t)$ $\varphi^{-1}(t) \neq \emptyset$ because φ is surjective

$t \neq t' \Rightarrow \varphi^{-1}(t) \cap \varphi^{-1}(t') = \emptyset$

1 \Rightarrow 2: partition $S = \bigcup_{i \in I} S_i \iff a \sim b \Leftrightarrow a, b \in \text{same block}$

2 \Rightarrow 3: given \sim on S , let $T = \{\text{equivalence classes}\}$

$\psi: S \rightarrow T$

$s \mapsto [s]$ other notations: \bar{s} , C_s

General e.g. G group $a \sim b$ if $gag^{-1} = b$ for some $g \in G$ conjugacy classes

check $a \sim a: a = eae^{-1}$

compare: similar matrices

$a \sim b \Rightarrow b \sim a: a = g^{-1}bg$

$a \sim b$ and $b \sim c \Rightarrow a \sim c: hbh^{-1} = c \Rightarrow (hg)a(hg)^{-1} = h(gag^{-1})h^{-1} = hbh^{-1} = c.$ ✓

E.g. $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$

$$(12)(123)(12) = (132)$$

partition of S_3 by cycle type!

conjugacy classes		
(1)(2)(3)	(12)	(123)
	(23)	(132)
	(13)	

$$(13)(12)(13) = (1)(23)$$

$$(23)(12)(23) = (13)(2)$$

$$(123)(12)(123)^{-1} = (1)(23)$$

$$(132)(12)(132)^{-1} = (13)(2)$$

Remark: Can use $S \rightarrow T$ not surjective, but must omit empty fibers from partition and \sim

General e.g. $\Psi: G \rightarrow G'$ $N = \ker \Psi$ $\Psi(a) = \Psi(b) \Leftrightarrow aN = bN$

Nonempty fibers all have form aN for some $a \in G$ $\Leftrightarrow b \in aN \stackrel{\text{def}}{=} \{an \mid n \in N\}$

Cosets

Fix subgroup $H \leq G$

Def: A left coset of H (in G) is a subset of G having the form aH for some $a \in G$.

Write $b \equiv a$ (b is congruent to a) if $b \in aH$ (i.e. $b = ah$ for some $h \in H$).

Prop: The left cosets of H partition G .

Pf: $a = ae$ and $e \in H$ ($a \equiv a$)

$$b = ah \Rightarrow a = b\underbrace{h^{-1}}_{\in H} \quad (a \equiv b \Rightarrow b \equiv a)$$

$$b = ah \text{ and } c = bh' \Rightarrow c = a\underbrace{h h'}_{\in H}. \quad \square$$

Cor: $aH \cap bH \neq \emptyset \Leftrightarrow aH = bH. \quad \square$

Def: $G/H = \{\text{cosets of } H \text{ in } G\}$, so

$G \rightarrow G/H$ has fibers aH for $a \in G$.

E.g. Another way to see S_3

$$e \quad (12) \quad (23) \quad (13) \quad (123) \quad (132)$$

$$1 \quad x \quad y \quad \begin{matrix} xyx^{-1} \\ yxy \end{matrix} \quad xy \quad yx$$

$$S_3 = \langle x, y \mid x^2 = 1, y^2 = 1, xyx = yxy \rangle$$

$$\begin{array}{c} \parallel \\ \text{generators} \end{array} \quad \begin{array}{c} \nwarrow \\ \text{relations} \end{array}$$

$$H = \langle x \rangle = \{1, x\}$$

$$yH = \{y, yx\}$$

$$xyH = \{xy, xxy\}$$

$$\left. \begin{array}{c} \\ \end{array} \right\} S_3/H$$

E.g. $G = \mathbb{Z}$ $H = 2\mathbb{Z}$ $G/H = \mathbb{Z}/2\mathbb{Z} = \{2\mathbb{Z}, 1+2\mathbb{Z}\} = \{\bar{0}, \bar{1}\}$
 even odds

$H = 3\mathbb{Z}$ $G/H = \mathbb{Z}/3\mathbb{Z} = \{3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\} = \{\bar{0}, \bar{1}, \bar{2}\}$